1 Bernstein and Lunts' Fundamental Example

1.1 Introduction

Let $\mathbb{C}^*$ operate on $\mathbb{C}^n$ in the natural way and let $X \subset \mathbb{C}^n$ be a $\mathbb{C}^*$-stable closed subvariety. Because X is $\mathbb{C}^*$-stable and closed the origin is contained in X. It is a interesting and important problem to study the topology of X. Using the $\mathbb{C}^*$-action it is easy to see that the origin is a deformation retract of X and hence ordinary homology and cohomology are uninteresting invariants of X. Therefore in order to study, for example how singular X is, we need more sophisticated machinery. One such tool is equivariant intersection cohomology.

We now want to sketch how Bernstein and Lunts' approach this situation and what they are able to prove. Let $\mathscr{C} = IC_{\mathbb{C}^*}(X)$ be the equivariant intersection cohomology complex on X. Setting $X_0 = X \setminus 0$ we have a diagram of topological spaces:

$$\{0\} \xrightarrow{i} X \xleftarrow{j} X_0$$

And hence an exact triangle in $D^b_G(X)$:

$$i_! i^! \mathscr{C} \longrightarrow \mathscr{C} \longrightarrow j_* j^* \mathscr{C} \xrightarrow{[1]}$$

Pushing everything onto a point, we then obtain an exact triangle in $D^b_G(pt)$:

$$i^! \mathscr{C} \longrightarrow p_* \mathscr{C} \longrightarrow p_* j^* \mathscr{C} \xrightarrow{[1]}$$

One of the central results of Bernstein and Lunts' book [2] is that $D^b_G(pt)$ has an elegant description (via taking Hom with the contant sheaf) as a triangulated category of differential graded modules over a differential graded algebra A_G. In our case (with $G = \mathbb{C}^*$) the differential graded algebra A_G is particularly simple: it is a polynomial ring in one variable of degree 2.

Bernstein and Lunts also contruct a t-structure on the triangulated category of differential graded modules. We are now able to state their theorem (albeit in slightly diluted form):

Theorem 1.1.1. *Let M be a differential graded module over $A_{\mathbb{C}^*}$ which corresponds to $p_* j^* \mathscr{C}$ under the correspondence of Bernstein and Lunts. Then the above triangle in $D^b_{\mathbb{C}^*}(pt)$ is isomorphic to the following triangle when viewed as an triangle of differential graded modules:*

$$\tau_{\geq} M[-1] \longrightarrow \tau_{<0} M \longrightarrow M \xrightarrow{[1]}$$

Moreover, it is possible to write $\tau_{\geq} M[-1]$ and $\tau_{<0} M$ as free $A_{\mathbb{C}^*}$-modules with trivial differential.

The objects which appear in the last triangle are explicitely computable and hence one can obtain a feel for the beauty of Bernstein and Lunts' treatment.

The aim of this article is to give an explantion of this result. I assume that the reader has some knowledge of the derived category of equivariant sheaves as well as some experience with non-equivariant perverse sheaves (although I hope to give a quick explanation of both). With this I am then able to give an explanation of the above result which I hope will fill in some gaps in understanding which may be present when on reads the proof of Bernstein and Lunts. I also hope that the introduction of weights into the derived category of equivariant sheaves (which was suggested by the article of Bradon and MacPherson [3]) makes some of the proofs structurally clearer (this has at least been my experience).

Unfortunately our explanation is far from complete. We assume the existence of a 'category with weights' as well as the statement of the hard Lefschetz theroem.

1.2 Fast and Furious G-Bundles

Before we begin describing Bernstein and Lunts' category we need to recall some concepts related to bundles.

1.2.1 Principle and Universal Bundles

We recall quickly the notion of a characteristic class. Let G be a topological group. A *principle G-bundle* is a map of spaces:

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ B \end{tikzcd}$$

where E is a G-space and π looks locally like the first projection $E' \times G \to E'$ (with the G-action trivial on E'). One checks that in a diagram

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ B' \arrow[r] & B \end{tikzcd}$$

the topological pullback is naturally a principle G-bundle.

The crucial observation is then that *bundles tend to get topologically simpler when one pulls back*. One is then led to suspecting the presence of a *universal G-bundle*. That is a bundle $\pi : EG \to BG$ so that *all* (paracompact) principle G-bundles can be obtained from the universal bundle by pulling back. It turns out that one can be even more picky and require that the map $f : B \to BG$ that gives the bundle via pull-back be unique up to homotopy. It is an amazing fact that for any topological group G a universal bundle exists:

Theorem 1.2.2. *For any topological group G a universal bundle $EG \to BG$ exists.*

For the simple construction (due to Milnor) of such a bundle for any topological group G see [7]. When G is closed subgroup of GL_n one can take a direct limit of Stiefel mannifolds (see Brion [5]). We will sketch the case of $G = \mathbb{C}^*$ below. We will need the following simple observation later:

Lemma 1.2.3. *Let $EG \to BG$ be a universal bundle. Then all the homotopy groups of EG vanish.*

Proof. A rather rough way of seeing this is that two non-homotopic maps $f, g : X \to EG$ would give two non-homotopic ways of getting the trivial bundle $X \times G \to X$ which is forbidden in the definition. □

.

We will now construct a universal $\mathbb{C}^*$-bundle. For all n the 'tautological bundle' $\mathbb{C}^n \setminus 0 \to \mathbb{P}^{n-1}\mathbb{C}$ ist naturally a principle $\mathbb{C}^*$-bundle. Taking the direct limit as topological spaces gives $\mathbb{C}^\infty \setminus 0 \to \mathbb{P}^\infty\mathbb{C}$. The following theorem states that we have found a universal $\mathbb{C}^*$-bundle.

Theorem 1.2.4. *The principle $\mathbb{C}^*$-bundle $\mathbb{C}^\infty \setminus 0 \to \mathbb{P}^\infty\mathbb{C}$ is universal. In other words, for any principle $\mathbb{C}^*$-bundle $\pi : E \to B$ with B paracompact there exists $f : B \to \mathbb{P}^\infty\mathbb{C}$ so that π is isomorphic to the bundle obtained by pulling back via f. Moreover, f is unique up to homotopy.*

Proof. For all the details see [7]. We will be happy to give a quick sketch. One first observes that *any* map of bundles

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathbb{C}^\infty \setminus 0 \arrow[d, "\pi"] \\ B \arrow[r, "f"] & \mathbb{P}^\infty\mathbb{C} \end{tikzcd}$$

gives a Cartesian diagram. Hence, to find a map f whose pull back is π it is enough to find a pair of horizontal maps above so that the diagram commutes. (Another way of saying this is that any map of bundles over a fixed base space is an isomorphism). Now there exists an open covering $\{U_i\}_{i \in S}$ over which the bundle is trivial and, because B is paracompact, there exists a partition of unity subordinate to $\{U_i\}$. In fact, one can assume that this partition if countable (this is a trick that I will let Husemoller explain!). It is then a simple matter to construct a $\mathbb{C}^*$-equivariant map $E \to \mathbb{C}^\infty \setminus 0$ which induces the desired map of bundles.

To see that any two such maps are homotopic one notices that is is possible to homotopically slide any two bundle maps so that the image only lives in the even or odd terms of $\mathbb{C}^\infty \setminus 0$ and $\mathbb{P}^\infty(\mathbb{C})$. Given two such maps one slides one map into the even terms, the other into the odd and then one can then easily write down a linear homotopy between them. □

1.2.5 Chern classes for priniple $\mathbb{C}^*$-bundles

We now return to general theory. Suppose that we have a group G and have constructed, as we did above for $\mathbb{C}^*$, a universal bundle $\pi : EG \to BG$. Then the definition of a universal bundle ensures that, for a given space B, principle G-bundles on B are in canonical bijection with homotopy classes of maps $B \to BG$. Suppose further that we know a set of generators for $H^*(BG;\mathbb{Z})$. Then, for any (homotopy class of a) map $f : B \to BG$ we obtain canonically elements of the cohomology of B by pulling back the generators of $H^*(BG;\mathbb{Z})$. But, by the above discussion such a collection of elements in $H^*(B;\mathbb{Z})$ can be viewed as invariants of the G-bundle f^*BG. This is the point of departure for the theory of "characteristic classes": the image of each generator of $H^*(BG)$ gives a "characterstic class" of the bundle in $H^*(B)$ and one can study to what extent these classes determine the bundle.

Let us restrict ourselves to the simplest case of $G = \mathbb{C}^*$. Then, for any $\mathbb{C}^*$-bundle $\pi : E \to B$ we can find a map $f : B \to \mathbb{P}^\infty\mathbb{C}$ so that the bundle obtained by pulling back is isomorphic to π. Because $H^*(\mathbb{P}^\infty\mathbb{C});\mathbb{Z})$ is isomorphic to $\mathbb{Z}[T]$ (with $\deg T = 2$) the only invariant of π that we obtain in $H^*(B)$ is the image of T under $f^* : H^2(\mathbb{P}^\infty\mathbb{C}) \to H^2(B)$. We will call this image the *Chern class* of the bundle π. We have the following amazing theorem (of Chern? Reference?):

Theorem 1.2.6. *Let B be a paracompact space. Then the Chern class of a $\mathbb{C}^*$-bundle $\pi : E \to B$ completely classfies π. Moreoever, for every element of $\gamma \in H^2(B;\mathbb{Z})$ there is a $\mathbb{C}^*$-bundle over B having γ as its Chern class. Hence, $\mathbb{C}^*$-bundles over B are in canonical bijection with $H^2(B;\mathbb{Z})$.*

1.3 The Equivariant Derived Category

For the rest of this work we will be dealing with $D^b_G(X)$ – the "derived category of G-equivariant sheaves on X". However, when one hasn't met the formalism of Bernstein-Lunts [2] or one has never worked in a derived category, the theory can be overwhelming and one can quickly loses sight of what one really wants! Hence we want to spend a few paragraphs motivating the concept of an equivariant sheaf and show why one needs the formalism of Bernstein-Lunts in order to get the category that one wants.

1.3.1 What is an equivariant sheaf?

Almost every "topological object with extra structure" can be viewed as a pair $(X, \mathscr{O})$ where X is a topological space and $\mathscr{O}$ is a sheaf. For example, a differentiable manifold is a Hausdorff topological space M together with a sheaf so that, for every point p, there exists a neighbourhood U of p so that $\mathscr{O}(U)$ looks like the C^∞-functions on an open set of some $\mathbb{R}^n$. Similarly one can define a topological or complex manifold or an algebraic variety or scheme. Hence, for a topological space X a sheaf on X formalises the "extra structure" that X may have.

When one first hears the defition of a sheaf one is usually told to think of the sections over an open set as "functions" on this set: it is precisely the requirement that such sections should behave like functions which distinguishes a sheaf from a presheaf. The equivalent notion for the sections over a set U of a G-equivariant sheaf are G-invariant functions on U. So a G-equivariant sheaf $\mathscr{F}$ on a G-space X should be a sheaf such that the sections over an open set are "G-invariant". Note that a set of continuous functions from X into a space is G-invariant precisely when the function space admits an G-equivariant action of G. For sheaves the analogy of the space of functions is the étale space of the sheaf (see [10]). Hence we are led to the following definition:

Definition 1.3.2. *A G-equivariant sheaf (or sets) on a G-space X is a sheaf $\mathscr{F}$ on X together with an equivariant action of G on the étale space $\overline{\mathscr{F}}$.*

Note: The one problem with this definition is that it completely falls apart in the algebraic situation (for reasons which I don't yet understand!). In this situation one has two maps from $G \times X$ to X: one can either operate with G or project. One then defines a G-equivariant sheaf on X to be a sheaf together with an isomorphism between the two sheaves obtained by pulling back along the operation or the projection (together with compatibility conditions). For discussion in this direction see the fantastic book [6] of Chriss and Ginzburg. For a (poor) explanation of why this leads to the above definition in the topological situation see [10].

Now, let us look at some examples of equivariant sheaves:

Equivariant sheaves on a point: Assume that X is a point. Because an étale space over a point is simply a discreet set, a G-equivariant sheaf is an action of G on a discreet space. If G is locally connected then this action factorises as an action of G/G^0 on a discreet space (where G^0 is the identity component). Hence, if G is locally connected we have an equivalence of categories:

$$\{\ G\text{-equivariant sheaves on a point}\} \xrightarrow{\sim} \{\text{sets with an action of } G/G^0\}$$

When the group acts freely: Suppose X is a G-space and G operates topologically freely on X. That means, every point in X has an open neighbourhood that looks like $G \times U$ with the operation of G only on the first factor (so we can think of the set U as a 'normal cross-section' to the action of G). Then it is a theorem (which is obvious if one restricts oneself to the case of continuous invariant functions) that a sheaf on G-equivariant sheaf on X is the same as a sheaf on X/G. Hence we have an equivalence of categories (see [10]):

$$\{\ G\text{-equivariant sheaves on a free space } X\} \xrightarrow{\sim} \{\ \text{sheaves on } X/G\}$$

Equivariant sheaves are harder than sheaves on the quotient space: After the example above one might think that every G-equivairant sheaf on X is the same as a sheaf on the quotient space. This is not the case, as can be seen when one considers $G = \mathbb{R}_{>0}$ acting on $X = \mathbb{R}_{\geq 0}$. Then the quotient space X/G consists of two points and four open sets (draw it!). In particular, the category of sheaves on X/G is not equivalent to sheaves on a point. However one can show that any G-equivariant sheaf on X is a constant sheaf. Hence the category is equivalent to the category of sheaves on a point.

One can have fun detemining the category of sheaves on various simple G-spaces. See [10] for more examples. In particular, one might like to think about the following example. Let $\mathbb{R}$ operate on the circle S^1 through rotation. What are the equivariant sheaves on S^1 with respect to this action?

1.3.3 Why do we want a derived category?

Let us forget that G operates on X and turn our minds back to normal sheaves on X. We have already convinced ourselves that sheaves really are the correct language to talk about spaces. We now come to two major insights of Gröthendieck that are still (at least for the author) a little mysterious. The first is that one should not restrict attention to sheaves, but should also consider *complexes* of sheaves.[1] The second is to realise that if we have two complexes of sheaves $\mathscr{F}$ and $\mathscr{G}$ and a map between then such that the the induced map $\mathcal{H}(\mathscr{F}) \to \mathcal{H}(\mathscr{G})$ is an isomorphism, then $\mathscr{F}$ and $\mathscr{G}$ should be seen as the *same* sheaves – this is, in effect, what happens when we go to the derived category.

Gröthendieck then showed, that when one makes these two assumptions the only "natural" extentions of the functors f^*, $f_!$, f_* etc. to derived categorys yields naturally, and gives a richer interpretation to, the techniques of algebraic topology and homological algebra. For example, if $p : X \to \mathrm{pt}$ is the projection to a point and $\underline{X}$ is the constant sheaf on X, then $p_*\underline{X}$ 'is' the cohomology of X. Moreover, many deeper results in algebraic topology, for example Poincarè duality, become formal consequences of relations between functors (after the work of Gröthendieck's student Verdier).

In this way one slowly becomes convinced the the derived category is really the right place in which to do algebraic topology. In fact (and this will be important below) we should see the derived category as a categorification of cohomology: in cohomology, for each map $f : X \to Y$, we get two graded algebras and a homomorphism between them; in Gröthendieck's approach, for each map $f : X \to Y$ we get two categories, $D(X)$ and $D(Y)$ and functors between them.

1.3.4 What is equivariant cohomology?

Recall from Section 1.2.1 that for any topological group G there exists a universal bundle $EG \to BG$. It is a simple consequence of what it means to be universal that EG is unique up to G-equivariant homotopy. If X is a G-space we let G operate diagonally on $X \times EG$ and denote the quotient space under this action by $X \times_G EG$. We now define the *G-equivariant cohomology*, which we will denote by $H^*_G(X)$, to be the cohomology of the quotient space $X \times_G EG$. Note that, since EG is unique up to equivariant homotopy, $H^*_G(X)$ is well defined.

[1] That is, a family $\{\mathscr{F}_i\}$ of sheaves and differentials $d_i : \mathscr{F}_i \to \mathscr{F}_{i+1}$ so that $d_{i+1}d_i = 0$.

Note that the map $X \to pt$ induces $X \times_G EG \to BG$ $(= pt \times_G EG)$ and hence the equivariant cohomology of X is always a module over the equivariant cohomology of a point; which in turn is the cohomology of BG. Hence, when dealing with a group G, the first step is usually to work out $H_G^*(pt) = H^*(BG)$. As seen in Section 1.2.5, the calculation of $H^*(BG)$ is also vital to the definition of characteristic classes. Since the theory of characteristic classes was developed before that of equivariant cohomology the theory gets a head start! We have, for example, the following result (see, for example, Brion [5]):

Theorem 1.3.5. *Let G be a reductive linear algebraic group over $\mathbb{C}$ with maximal torus T and Weyl group W and let S be the group of characters of T. Then $H_G^*(pt)$ is equal to the W-invariant polynomial functions on $T \otimes_{\mathbb{Z}} \mathbb{Q}$, with the coordinate function of each element of S having degree 2.*

Hence, for example, $H_{\mathbb{C}^*}^*(pt) = \mathbb{C}[T]$ with T having degree 2. (This is also clear from the fact that $B\mathbb{C}^* = \mathbb{P}^\infty\mathbb{C}$).

Note that, if X is a free G-space, then $X \times_G EG \to G \setminus X$ is a fibration with fibre EG. However, we have seen in Lemma 1.2.3 that EG is always contractible and hence, if X is a free G-space we have $H_G^*(X) = H^*(G \setminus X)$. This should remind the reader of the corresponding result for equivariant sheaves.

Up until now we have said very little about what equivariant cohomology really *is*. To explain this we need a little more formalism. We say that a map $f : P \to X$ is ∞-*acyclic*, if for all complexes of sheaves $\mathscr{F}$ on X, $f_* f^* \mathscr{F}$ and $\mathscr{F}$ are isomorphic and that this property holds if we replace f by f' in a Cartesian diagram:

$$\begin{tikzcd} P' \arrow[r] \arrow[d, "f'"] & P \arrow[d, "f"] \\ X' \arrow[r] & X \end{tikzcd}$$

[In modern language: we require that the adjunction morphism $id \to f_* f^*$ is an isomoprhism and that this property be stable under base change.[2]]

Now let $f : P \to X$ be an ∞-acyclic map and $\underline{X}$ and $\underline{P}$ the constant sheaves on X and Y respectively. Since f must be surjective we have $f^*\underline{X} = \underline{P}$ and the fact that f is ∞-acyclic implies that $f_* f^* \underline{X}$ and $\underline{X}$ are isomorphic. Hence, in particular, $H^*(P)$ and $H^*(X)$ are isomorphic. This shows that ∞-acyclic maps are very special indeed!

We have the following result which characterises ∞-acyclic maps (it is quoted in Bernstein-Lunts [2] as a variant of the Vietoris-Begle theorem.):

Lemma 1.3.6. *A map $f : P \to X$ is ∞-acyclic if it is a topological fibration and the fibres have trivial cohomology.*

We now attempt to motivate equivariant cohomology a little. Let X by a G-space. Equivariant cohomology is, very roughly, the cohomology of the quotient

[2]This is another 'Gröthendieck philosophy': most properties worth considering are stable under base change!

space X/G. However, simply taking the cohomology of X/G is too rough for two reasons. The first is that X/G is rarely a well-behaved space and hence one cannot expect taking cohomology to be meaninful. The second reason is that, even if X/G is well-behaved (for example, a manifold) we have still lose too much information on the operation of G on X when we pass to X/G. For example, if G operates on a point then, in passing to the quotient space (which is again a point) we lose all information about G.

However, if G operates topologically freely on X then we can define $H_G^*(X) := H^*(X/G)$. Otherwise we 'resolve X' by finding a ∞-acyclic map $f : P \to X$ so that G operates topologically freely on P, and then define $H_G^*(X) := H^*(P/G)$. At this point it is a nice exercise (using the fact that ∞-acyclic maps are stable under base change) to show that, with this alternative defintion, $H_G^*(X)$ does not depend on the choice ∞-acyclic resolution.

To see that the two definitions are equivalent notice that EG is a topologically free G-space (since it is a principle bundle) and is contractible (Lemma 1.2.3). Hence, by Lemma 1.3.6 above, $EG \to pt$ is ∞-acyclic. Hence, by the definition of ∞-acyclic, for any space X the first projection $X \times EG \to X$ is a ∞-acyclic. Hence $H_G^*(X) = H^*(X \times_G EG)$ which recovers our earlier definition. (Note that this also shows that a ∞-acyclic resolution of a G-space X *always* exists – a fact that is initially not clear. This is known as the *Borel construction*).

1.3.7 Bernstein and Lunt's Category

We have now seen the three skeletons of what we want the 'equivariant derived category' to be: we want it to be an 'extension'[3] of the category of equivariant sheaves and we want that it is a 'categorification' of equivariant cohomology, in the same way that the derived category is a categorification of normal cohomology.

We might naively define $D_G^+(X)$ to be the derived category of equivariant sheaves — that is, an element of $D_G^+(X)$ should be a complex of equivariant sheaves (up to quasi-isomorphsim). However, simple examples show that such a definition is silly. For example, consider the group S^1 and the two S^1-spaces S^1 itself and a point. Then, after what we have seen in Section 1.3.1 the categories of equivariant sheaves on these two spaces are equivalent, and hence their derived categories are also equivalent. However we would like our categories to be able to tell the difference between a point and a circle!

As in the case of equivariant cohomology, it is clear how we should define our category if G operates freely on X. For, as we mentioned in Section 1.3.1, we have an equivalence (where $\pi : X \to X/G$ is the projection):

$$\begin{array}{ccc} \{\ G\text{-equivariant sheaves on } X\} & \xrightarrow{\sim} & \{\ \text{sheaves on } X/G\} \\ \pi^*\mathscr{F} & \longleftarrow\!\!\!\shortmid & \mathscr{F} \end{array}$$

[3]More precisely: we want a t-structure on our category so that the heart is the category of equivariant sheaves.

And hence it is natural to define $D^+_G(X)$ to be the derived category $D^+(X/G)$. Now, if G does not operate topologically freely on X we have seen that we can always find a ∞-acyclic map of G-spaces $f : P \to X$. Then, using this map (or rather the functors f^* and f_*) we can identify $D^+(X)$ with a full subcategory of $D^+(P)$. The advantage here is that we know which objects in $D^+(P)$ are G-equivariant – they are precisely the objects $\mathscr{F}$ that 'come from' P/G in the sense that there exists $\mathscr{G} \in D^+(P/G)$ so that $\pi^*\mathscr{G} = \mathscr{P}$ (where $\pi : P \to P/G$ is the quotient map). Hence we are led to the following definition:

Definition 1.3.8. *Let X be a G-space and choose a ∞-acyclic G-map $f : P \to X$ so that G operates freely on P and let $\pi : P \to P/G$ be the quotient map. Then objects in $D^+_G(X)$ consist of triples $(\mathscr{F}, \mathscr{G}, \alpha)$ where $\mathscr{F} \in D^+(X)$, $\mathscr{G} \in D^+(P/G)$ and α is an isomorphism $f^*\mathscr{F} \to \pi^*\mathscr{G}$. A morphism between two objects $(\mathscr{F}, \mathscr{G}, \alpha)$ and $(\mathscr{F}', \mathscr{G}', \alpha')$ is a pair of morphisms $b : \mathscr{F} \to \mathscr{F}'$ and $q : \mathscr{G} \to \mathscr{G}'$ so that the following diagram in $D^+(P)$ commutes:*

$$\begin{tikzcd} \pi^*\mathscr{G} \arrow[r, "\pi^* q"] & \pi^*\mathscr{G}' \\ f^*\mathscr{G} \arrow[u, "\alpha"] \arrow[r, "f^* q"] & f^*\mathscr{G}' \arrow[u, "\alpha'"'] \end{tikzcd}$$

Again one can show using base change that different choices of ∞-acyclic resolutions $f : P \to X$ yield equivalent categories. Note also, that if G already operates topologically freely on X then we can choose $id : X \to X$ as our ∞-acyclic resolution and recover the equivalence $D^+_G(X) \xrightarrow{\sim} D^+(X/G)$. Lastly, replacing D^+ with D^b throughout the above definition we obtain the bounded equivariant derived category which we will denote $D^b_G(X)$.

It is difficult to describe $D^+_G(X)$ or $D^b_G(X)$ in most cases. In fact, it is an area of current research to describe $D^b_G(X)$ when G has finitely many orbits on X. However, if we restrict ourselves to the case of $X = pt$ we have the following nice description given Bernstein and Lunts [2]:

Theorem 1.3.9. *The category $D^b_G(pt)$ is equivalent to the subcategory of $D^b(BG)$ consisting of those complexes which locally constant cohomology sheaves.*

At this point we will leave a further description of the general equivariant derived category (in particular the definition of functors) to Bernstein and Lunts' book [2]. I hope that at this point the reader feels motivated to learn more — this has at least been my experience!

1.4 The equivariant category in the algebraic situation

We now want to leave the topological situation and specialise to the case of complex algebraic varieties. This immediately introduces two problems which we will now attempt to describe. In the topological case if X is a G-space then X/G *always* exists. Moreover, if the action of G on X is topologically free then

$X \to X/G$ is a topological fibration. Niether of these statements remain true for varieties. We now give a definition which defines a class of spaces for which the above two properties hold.

Definition 1.4.1. *Let X be a complex algebraic variety, acted on by a linear algebraic group G. A* resolution *of X is a map $f : P \to X$ of G-varieties so that: (i) f is a Zariski locally trivial fibration with smooth fibres; (ii) G acts Zariski freely on P.*

Note that, as a consequence of (ii), if $f : P \to X$ is a resolution then P/G exists and the map $P \to P/G$ is a Zariski locally trivial fibration.

The second problem is that, in the definition of $D^b_G(X)$ it was vital to be able to choose a ∞-acyclic map of G-spaces $f : P \to X$ so that G operates freely on P. However, even when $G = \mathbb{C}^*$ and $X = pt$ we have seen thata natural choice $P = \mathbb{C}^\infty \setminus 0$ is infinite dimensional and therefore is not a variety. It is in fact easy to see (because $H^*_{\mathbb{C}^*}(pt) = \mathbb{C}[T]$ is infinite dimensional) that it is not possible to find a $\mathbb{C}^*$-variety P and a map $f : P \to pt$ that is ∞-acyclic.

We conclude that, in the category of G-varieties, where G is a linear algebraic group, there are not enough ∞-acyclic maps. In the next section we will see how to repair this problem!

1.4.2 What happens when there are not enough ∞-acyclic maps?

The philosophy of how one constructs $D^b_G(X)$ when there are not enough ∞-acyclic resolutions exist is easy: one constructs 'approximations' to $D^b_G(X)$ and then views $D^b_G(X)$ as a 'limit' of these approximations.[4] However in practice the definition is a little tedious. Bernstein and Lunts [2] give two equivalent definitions. We will work with the second, which we feel is more appropriate to the algebraic situation:

Definition 1.4.3. *Let X be a complex algebraic variety acted on by a linear algebraic group G.*

An object $\mathscr{F} \in D^b_G(X)$ is an association which assigns to each resolution $f : P \to X$ an object $\mathscr{F}(P) \in D^b(P/G)$ and each map of resolutions $\alpha : P \to Q$ over X an isomorphism $\mathscr{F}(\alpha) : \mathscr{F}(P) \to \overline{\alpha}^*\mathscr{F}(Q)$ (where $\overline{\alpha} : P/G \to Q/G$ is the quotient map).

A morphism between two objects $\mathscr{F}$ and $\mathscr{G}$ in $D^b_G(X)$ is a morphism between $\mathscr{F}(P) \to \mathscr{G}(P)$ for every resolution $f : P \to X$ so that the obvious diagram commutes.

Let us forget quickly the adjective 'resolution' in the definition and consider what this definition means in terms of the earlier topological situation. Here a ∞-acyclic map of G-spaces $P \to X$ with P a topologically free G-space always

[4] The astute reader will have noticed that have only mentioned the construction of the bounded equivariant derived category. This is because it is only possible to construct the unbounded category $D^+_G(X)$ when a ∞-acyclic map exists.

exists and for any resolution $Q \to X$ we have a Cartesian square:

$$\begin{tikzcd} Q \times_X P \arrow[r, "\alpha"] \arrow[d, "\beta"] & P \arrow[d] \\ Q \arrow[r] & X \end{tikzcd}$$

Now, if $\mathscr{F}$ is an object in $D^b_G(X)$ then we have isomorphisms $\overline{\beta}^* \mathscr{F}(Q) \to \mathscr{F}(Q \times_X P) \to \overline{\alpha}^* \mathscr{F}(P)$. Now, by base change, $\overline{\beta}$ is ∞-acyclic and hence, applying $\overline{\beta}_*$ to the above isomorphisms we obtain that $\mathscr{F}(Q) \to \overline{\beta}_* \overline{\alpha}^* \mathscr{F}(P)$ is an isomorphism. In other words, if an ∞-acyclic resolution $f : P \to X$ exists then the complex $\mathscr{F}(P) \in D^b(P/G)$ determines $\mathscr{F}$. Thus we recover our earlier definition (which should be reassuring for the reader!).

1.4.4 A technical point on resolutions

In the definition of the previous section we have ignored a subtle point which we now want to mention. Notice that, in the definition of $D^b_G(X)$ for a G-variety X we have not considered all G-maps $f : P \to X$ with the G-operation on P topologically free, rather we have restricted ourselves to particularly nice maps which we called 'resolutions'. The problem here is that there might not be enough resolutions for our above definition to yield the right category!

To give a little more detail we first need the notion of an n-acyclic map. For this, let $D^b(X)$ be the bounded derived category and if $I \subset \mathbb{Z}$ is an interval, let $D^I(X)$ denote the full subcategory of complexes whose cohomology sheaves are non-zero only for $i \in I$. Now let $I = [0, n]$. A map $f : P \to X$ is called *n-acyclic* if, for all sheaves $\mathscr{F} \in D^I(X)$, the natural morphism $\mathscr{F} \to \tau_{\leq n} f_* f^* \mathscr{F}$ is an isomorphism. The reader can check that a map $f : P \to X$ is ∞-acyclic if and only if it is n-acyclic for all n. Bernstein and Lunts then give precise conditions on a family of maps $f : P \to X$ so that the definition of the previous section gives the 'right' category. In our situation we can express these conditions in the following:

Lemma 1.4.5. *Let X be a G-variety. Assume that for all n there exists an n-acyclic resolution $f : P \to X$. Then the algebraic definition of $D^b_G(X)$ is equivalent to the topological definition of $D^b_G(X)$ (that is, the definitions in 1.3.8 and 1.4.3 are equivalent).*

Now the existence of an n-acyclic resolution for a G-variety X follows via base change from the existence of such a resolution when $X = pt$. Hence the question is reduced to the following: given a linear algebraic group G, does there exist an n acyclic resolution of a point for all n. The standard construction of a GL_n bundle over a Grassmannian (see, for example, Chapter VI of Shafarevich [9]) gives a positive answer for GL_n:

Lemma 1.4.6. *For any n there exists an n-acyclic resolution of a point for $G = GL_n(\mathbb{C})$.*

However the existence of n-acyclic resolutions of a point for arbitrary linear algebraic groups remain a mystery for the author!

1.4.7 Equivariant Intersection Cohomology Complexes

Let us start by recalling what an intersection cohomology complex is. If X is a smooth complex projective variety the cohomology $H^*(X)$ satisfies a number of extremely deep properties: in particular Poincaré duality and the hard Lefschetz theorem. However, all of this theory completely falls apart when X is singular, and hence no longer a complex manifold. Goresky and MacPherson realised that the reason for the failure of Poincaré duality is that one cannot intersect cycles nicely on singular spaces. They then defined a cohomology theory for possibly singular varieties in which certain cycles (those for which intersections are not well behaved) are not allowed. There theory was a remarkable success and the corresonding cohomology groups are known as the 'intersection cohomology' of the space, and denoted $IH^*(X)$. (For an excellent introduction to intersection cohomology see the book edited by Borel [4] or the notes of Rietsch [8]).

Of course the next step was to take the combinatorial definition of intersection cohomology and interpret it sheaf theoretically; that is, just as the normal cohomology of a space if the cohomology of the constant sheaf, the intersection cohomology of a variety X should be the cohomology of an 'intersection cohomology complex' $IC(X)$. This result was achieved by Deligne who also gave two very elegant constructions of $IC(X)$. One of which we will now describe (for the other construction, and much more, see the book of Beilinson, Bernstein and Deligne [1]).

We first construct the so called *perverse t-structure*. Recall that a complex $\mathscr{F} \in D^b(X)$ is *constructible* if there exists a stratification $X = \sqcup_{S\in\mathcal{S}} S$ of X with smooth subvarieties so that $i_S^*\mathscr{F}$ is locally constant for each strata S (where $i_S : S \hookrightarrow X$ is the inclusion). We will denote the full subcategory of constructible complexes by $D_c^b(X)$. We make the following definitions:

$$
\begin{aligned}
{}^pD^{\leq 0}(X) &= \left\{ \begin{array}{c} \text{full subcategory of complexes } \mathscr{F} \in D_c^b(X) \\ \text{satisfying } \mathcal{H}^n(i_S^*\mathscr{F}) = 0 \text{ for } n > -\dim_{\mathbb{C}} S \end{array} \right\} \\
{}^pD^{\geq 0}(X) &= \left\{ \begin{array}{c} \text{full subcategory of complexes } \mathscr{F} \in D_c^b(X) \\ \text{satisfying } \mathcal{H}^n(i_S^!\mathscr{F}) = 0 \text{ for } n > -\dim_{\mathbb{C}} S \end{array} \right\}
\end{aligned}
$$

In both definitions a stratification $X = \sqcup_{S\in\mathcal{S}} S$ is chosen so as to make the $\mathscr{F} \in D_c^b(X)$ constructible and $i_S : S \hookrightarrow X$ denotes the inclusion.

The pair $({}^pD^{\leq 0}(X), {}^pD^{\geq 0}(X))$ give the *perverse t-structure* on $D_c^b(X)$ and, by general theory, the heart $Perv(X) :={}^p D^{\leq 0}(X) \cap {}^p D^{\geq 0}(X)$ is an abelian category. We also get a projection functor $\pi : D_c^b(X) \to Perv(X)$. (The analogy of course for this construction is the standard t-structure $(D^{\leq 0}(X), D^{\geq 0}(X))$ which gives the abelian category of sheaves sitting inside $D^b(X)$ together with the projection functor $\mathcal{H}^0$).

If X is smooth then $IC(X)$ should be a locally constant sheaf (traditionally sitting in degree $-d_X := -\dim_{\mathbb{C}} X$ – so that 0 is the "mirror" of Poincaré duality). Otherwise, one chooses a smooth open dense subvariety $j : U \hookrightarrow X$ and extends a local system $\mathscr{L}$ (again sitting in $-d_u := -\dim_{\mathbb{C}} U$) 'between' j_* and $j_!$ in the following sense. Because $j^* j_* \xrightarrow{\sim} id$ and $j^* = j^!$ there is an adjunction morphism $j_! \to j_*$.[5] We then have the following diagram (where α is the map given by the adjunction morphism):

$$\begin{tikzcd}
\text{in } Perv(U) & \text{in } D^b(X) & \text{in } Perv(X) \\
& j_!\mathscr{L}[d_U] \arrow[r, dashed] \arrow[d, "\alpha"] & \pi j_!\mathscr{L}[-d_U] \arrow[d, "\pi(\alpha)"] \\
\mathscr{L}[-d_u] \arrow[ur, dashed] \arrow[dr, dashed] & & \\
& j_*\mathscr{L}[d_U] \arrow[r, dashed] & \pi j_*\mathscr{L}[-d_U]
\end{tikzcd}$$

Now, $Perv(X)$ is an abelian category and hence we can define the *intermediate extension* of $\mathscr{L}[-d_u]$ denoted $j_{!*}$ to be the *image* of $\pi(\alpha)$. In fact, $j_{!*}$ is a functor from $Perv(U)$ to $Perv(X)$. We then define the *intersection cohomology complex extending* $\mathscr{L}$ to be $j_{!*}\mathscr{L}[d_U]$. We will not say any more in this general setting but rather refer the reader again to [1], [4] or [8].

In the equivariant situation one copies the construction above, adjusting to the added technicalities of the equivariant situation. We will describe this construction now in a little more detail than given in Bernstein and Lunts.

Our first task is to construct the abelian category of perverse sheaves sitting inside $D^b_G(X)$. For any complex variety Y let d_Y denote the complex dimension. If $f : P \to X$ is a resolution let $d_{P/X} = d_p - d_X$ denote the complex dimension of the fibres.

Definition 1.4.8. *We say that* $\mathscr{F} \in D^b_G(X)$ is perverse *if, for every resolution* $f : P \to X$*,* $\mathscr{F}(P)[d_{P/X} - d_G] \in D^b(P/G)$ *is perverse. The full subcategory of equivariant perverse sheaves on* X *will be denoted* $Perv_G(X)$.

Bernstein and Lunts define a forgetfull functor by $For : D^b_G(X) \to D^b(X)$ by $For(\mathscr{F}) := \mathscr{F}(X \times G) \in D^b(X \times_G G) \xrightarrow{\sim} D^b(X)$ (the last equivalence follows because $X \times_G G$ and X are canonically isomoprhic). Bernstein and Lunts then say that $\mathscr{F} \in D^b_G(X)$ is perverse if $For(\mathscr{F}) \in D^b(X)$ is. To get the equivalence between the two definitions notice that for any resolution $f : P \to X$ both maps $X \xleftarrow{f} P \xrightarrow{\pi} P/G$ are fibrations with smooth fibres. Hence the equivalence follows from the following lemma:

Lemma 1.4.9. *Let* $f : P \to X$ *be a locally trivial fibration with smooth fibres of dimension* $d_{P/X}$*. Then* $\mathscr{F} \in D^b(X)$ *is perverse if and only if* $f^*\mathscr{F}[d_{P/X}] \in D^b(P)$ *is perverse.*

Proof. The proof is straightforward using the condition of what it means to be perverse and the fact that it is possible to replace $f^!$ with $f^*[2d_{P/X}]$ if f is a

[5] This is just an extension of the fact that, before we consider derived functors, $j_!\mathscr{F}$ is a subsheaf of $j_*\mathscr{F}$.

fibration with smooth fibres (unfortunately I don't know a reference for this result (maybe Kashiwara?)). $\square$

One can easily show that $Perv_G(X)$ is the heart of a t-structure (by adjusting the earlier definition by the shifts above) and is therefore an abelian category (again by general theory). We also get a projection functor $\pi : D^b_G(X) \to Perv_G(X)$ which will be important in the definition of the equivariant intersection complexes.

We will now describe the construction of the equivariant intersection complexes which will parallel the above construction. Let X be a complex algebraic variety acted upon by a linear algebraic group G and let U be an open, dense, smooth and G-invariant subset of X and denote by $j : U \hookrightarrow X$ the inclusion (such a U exists: take, for example, all the smooth points). Then one can show that an equivariant perverse sheaf $\mathscr{G}$ on U is nothing other than a G-equivariant local system $\mathscr{L}$ on U shifted so as to sit in degree $-d_U$. Let $\mathscr{L}$ be such a perverse sheaf. Then, one can also show that $j^* j_*$ is naturally equivalent to the identity functor and that $j^!$ and j^* are isomorphic. Hence, exactly the same diagram above functions in the equivariant situation and we can define the *G-equivariant intersection cohomology complex extending $\mathscr{L}$*, denoted $IC_G(X, \mathscr{L})$ to be $j_{!*}\mathscr{L}$.

The above construction is very theoretical and we want to spend a few moments bgiving a sense for what these complexes are. Let X, U and $\mathscr{L}$ be as above and recall that for any resolution $g : V \to U$, $\mathscr{L}(V)$ is a local system on V/G. Now let $f : P \to X$ be a resolution and let $V = f^{-1}U$. Then V is also an open and dense subvariety of P. Denote by $j^{V/G}$ the inclusion of V/G in P/G. Then one can show (where $j_{!*}$ is the equivariant functor and $j_{!*}^{V/G}$ is the non-equivariant functor between $Perv(U)$ and $Perv(P)$):

Lemma 1.4.10. *We have:* $(j_{!*}\mathscr{L}[d_X])(P) = (j_{!*}^{V/G}\mathscr{L}(P)[d_{P/G}])[d_G - d_{P/X}]$.

The formula only looks complicated! For a proof one only needs to write out the equivariant functors $j_!$ and j_* and the definition of what it means to be perverse. We have the following nice consequence:

Lemma 1.4.11. *There is a natural identification:*

$$IC_G(X, \mathscr{L})(P) \xrightarrow{\sim} IC(P/G, \mathscr{L}(P))[d_G - d_{P/X}]$$

Hence the equivariant intersection cohomology complexes consist simply of a shifted intersection cohomology complex on each space P/G where $P \to X$ is a resolution; moreover, the local system which the complex extends is given by $\mathscr{L}(P)$.

1.5 The Fundamental Example

1.5.1 The Easiest Example

Before we consider the general question, let us consider the simplest case of the fundamental example. Let $\mathbb{C}^*$ operate on $X = \mathbb{C}^n$ in the natural way (by rescaling vectors). Then, following the notation in the introduction, $X_0 = \mathbb{C}^n \backslash 0$, and we have the following diagram of spaces:

$$\{0\} \xrightarrow{i} X \xleftarrow[j]{} X_0$$

Let $\mathscr{C}$ be the equivariant intersection cohomology complex $IC_{\mathbb{C}^*}(X)$ on X. In this case, X is smooth and so $\mathscr{C}$ is just the constant sheaf in degree $-n$ on each resolution of X (or, if we take the easier topological approach, the constant sheaf on $X \times_{\mathbb{C}^*} (\mathbb{C}^\infty \setminus 0)$). Let $\mathbb{Q}[T]$ be the $\mathbb{C}^*$-equivariant cohomology ring of a point. Then we have, as $\mathbb{Q}[T]$-differential graded modules):

$p_*\mathscr{C} = \mathbb{Q}[T][n]$: Because $\mathbb{C}^n \times_{\mathbb{C}^*} (\mathbb{C}^\infty \setminus 0) \to 0 \times_{\mathbb{C}^*} (\mathbb{C}^\infty \setminus 0) = \mathbb{P}^\infty\mathbb{C}$ is a homotopy equivalence.

$p_*j^*\mathscr{C} = \mathbb{Q}[T]/(T)[n]$: Because $(\mathbb{C}^\infty \setminus 0) \to X_0 \times_{\mathbb{C}^*} (\mathbb{C}^n \setminus 0) \to X_0/\mathbb{C}^* = \mathbb{P}^{n-1}\mathbb{C}$ is a fibration with contractible fibres.

$i^!\mathscr{C} = \mathbb{Q}[T][-n]$: Because, in this case $i^! = i^*[-2n]$.

All of this fits in the the following exact triangle:

	$i^!\mathscr{C} = \mathbb{Q}[T][-n]$ $\to$	$p_*\mathscr{C} = \mathbb{Q}[T][n]$ $\to$	$p_*j^*\mathscr{C} = \mathbb{Q}[T]/(T^n)[n]$ $\to$
$n+3$	0	0	0
$n+2$	$\mathbb{Q}T$	$\mathbb{Q}T^{n+1}$	0
$n+1$	0	0	0
n	$\mathbb{Q}$	$\mathbb{Q}T^n$	0
$n-1$		0	0
$n-2$		$\mathbb{Q}T^{n-1}$	$\mathbb{Q}T^{n-1}$
$\vdots$		$\vdots$	$\vdots$
$-n+2$		$\mathbb{Q}T$	$\mathbb{Q}T$
$-n+1$		0	0
$-n$		$\mathbb{Q}$	$\mathbb{Q}$

Where the first map is multiplication by T^n and the second if the projection (how to I show this in the diagram above?).

1.5.2 Equivariant Cohomology and the Lefschetz operator

Suppose that we are in the situation of the introduction: $\mathbb{C}^*$ operates on $\mathbb{C}^n$ in the natural way and $X \subset \mathbb{C}^n$ is a closed, $\mathbb{C}^*$-stable subvariety and $X_0 := X \setminus 0$. The fact that X_0 comes with an embedding in $\mathbb{C}^n \setminus 0$ gives us an embedding of the quotient $\overline{X} := \mathbb{C}^* \setminus X_0$ in $\mathbb{P}^{n-1}\mathbb{C}$ and hence, if we choose a generator of the cohomology of $\mathbb{P}^{n-1}\mathbb{C}$ we obtain, by pulling back via the embedding $\overline{X} \hookrightarrow \mathbb{P}^{n-1}\mathbb{C}$, an element of the cohomology of $\overline{X}$. This is the *Lefschetz operator* corresponding to the embedding $\overline{X} \hookrightarrow \mathbb{P}^{n-1}\mathbb{C}$. Writing out the definitions, and embedding $\mathbb{P}^{n-1}\mathbb{C}$ in $\mathbb{P}^{\infty}\mathbb{C}$, we see the the Lefschetz operator agrees with the Chern class of the principle $\mathbb{C}^*$-bundle $X_0 \to \overline{X}$.

We now want to relate the Lefschetz operator on

Proof. Consider the following diagram of spaces:

$$\begin{tikzcd}
\mathbb{C}^\infty \setminus 0 \arrow[d] & X_0 \times (\mathbb{C}^\infty \setminus 0) \arrow[l] \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & \mathbb{C}^\infty \setminus 0 \arrow[d] \\
\mathbb{P}^\infty\mathbb{C} & X_0 \times_{\mathbb{C}^*} (\mathbb{C}^\infty \setminus 0) \arrow[l, "\pi"'] \arrow[r, "f"] & \overline{X} \arrow[r, "i"] & \mathbb{P}^\infty\mathbb{C}
\end{tikzcd}$$

All squares are Cartesian and hence the two bundles that we obtain by pulling back $\mathbb{C}^\infty \to \mathbb{P}^\infty\mathbb{C}$ via π and $i \circ f$ are isomorphic. However $\mathbb{C}^\infty \to \mathbb{P}^\infty\mathbb{C}$ is a universal bundle and so π and $i \circ f$ are homotopic. Hence, if T is a generator for $H^*(\mathbb{P}^\infty\mathbb{C}; \mathbb{Q})$ then $\pi^*T = f^*i^*T$. The result then follows since i^*T is the Lefschetz operator and π^*T is the equivariant operator. $\square$

1.5.3 Schrott!

We first need the notion of an n-acyclic map. For this, let $D^b(X)$ be the bounded derived category (consisting of complexes $\mathscr{F}$ whose cohomology sheaves $\mathcal{H}^i(\mathscr{F})$ are non-zero for only finitely many i). If $I \subset \mathbb{Z}$ is an interval, let $D^I(X)$ denote the full subcategory of complexes whose cohomology sheaves are non-zero only for $i \in I$.

Definition 1.5.4. *Let $I = [0, n]$. A map $f : P \to X$ is called n-acyclic if, for all sheaves $\mathscr{F} \in D^I(X)$, the natural morphism $\mathscr{F} \to \tau_{\leq n} f_* f^* \mathscr{F}$ is an isomorphism.*

There are two simple consequences of the definition for a n-acyclic map $f : P \to X$. The first is that $H^*(P)$ and $H^*(X)$ agree up to degree n. The second is that if $I = [a, b]$ is an interval with $b - a \leq n$ then, for any $\mathscr{F} \in D^I(X)$ we have that $\mathscr{F} \to \tau_{\leq b} f_* f^* \mathscr{F}$ is an isomorphism.

Now, note that if $f : P \to X$ is n-acyclic of G-spaces, and G operates on P such that the quotient P/G exists as a variety we can recover a 'chunk' of the equivariant derived category by recycling the old argument: for any interval $I = [a, b] \subset \mathbb{Z}$ with $b - a \leq n$ we can use the functors f^* and $\tau_{\leq b} f_*$ to identify $D^I(X)$ with a subcategory of $D^I(P)$ — a category in which we know what it means to be equivariant. Hence, a G-equivariant object on X should be a complex $\mathscr{F} \in D^I(X)$, a complex $\mathscr{G} \in D^I(P/G)$ and an isomorphism α between $f^*\mathscr{F}$ and $\pi^*\mathscr{G}$ (where $\pi : P \to P/G$ is the projection).[6] We will temporarily denote this category by $D^I(X, P)$.

[6] We are doing nothing more than rehashing the definition from the previous section!

However, just as in the topological situation there are many candidates for an n-acyclic map $f : P \to X$. There is also the question of how the categories behave as one takes bigger and bigger intervals. The following lemma of Bernstein and Lunts [2] is fundamental:

Lemma 1.5.5. *Let $I = [a, b]$ and $J = [c, d]$ be an intervals with $I \subset J$ and let $f : P \to X$ and $g : Q \to X$ be n- and m-acyclic maps respectively. Assume further that $n > b - a$ and $m > d - c$. Then the categories $D^I(X, P)$ and $D^I(X, Q)$ are equivalent. Moreover, if $n > m$ there is a fully faithful functor embedding $D^I(X, P)$ in $D^J(X, Q)$.*

Hence, in order to define $D^I_G(X)$ we don't need to worry about the choice of map $f : P \to X$; for any interval $I \subset \mathbb{Z}$, we find an n-acyclic resolution $P \to X$ with n bigger than the length of I and define:

$$D^I_G(X) := D^I(X, P)$$

The second part of the lemma then allows us to define the equivariant bounded derived category $D^b_G(X)$ as the limit over all intervals $I \subset \mathbb{Z}$. As a formula:

$$D^b_G(X) := \lim_{I \subset Z} D^I_G(X)$$

One can of course show that if we use the above definition to define the topological category then one has an equivalence with a suitable bounded subcategory of $D_G(X)$.

References

[1] A. A. Beĭlinson, J. Bernstein and P. Deligne *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, **100**, Soc. Math. France, Paris, 1982.

[2] J. Bernstein and V. Lunts, *Equivariant Sheaves and Functors*, Springer Lecture Notes in Math. **1578** (1994), Springer.

[3] T. Bradon and R. MacPherson, *From moment graphs to intersection cohomology*, Math. Ann. **321** (2001), no. 3, 533–551.

[4] A. Borel, *Intersection Cohomology*, Boston, Birkhäuser, 1984.

[5] M. Brion, *Equivariant cohomology and equivariant intersection theory*, Notes by Alvaro Rittatore. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **514**, Representation theories and algebraic geometry (Montreal, PQ, 1997), 1–37, Kluwer Acad. Publ., Dordrecht, 1998.

[6] N. Chriss and V. Ginzburg, *Representation Theory and Complex Geometry*, Birkhäuser Boston, Inc., Boston, 1997.

[7] D. Husemoller, *Fibre Bundles*, Graduate Texts in Mathematics. **20** (1975), Springer, New York.

[8] K. Rietsch, *An introduction to perverse sheaves*, Representations of finite dimensional algebras and related topics in Lie theory and geometry, 391–429, Fields Inst. Commun., **40**, Amer. Math. Soc., Providence, RI, 2004.

[9] I. Shafarevich, *Basic algebraic geometry. 2. Schemes and complex manifolds*, Second edition. Translated from the 1988 Russian edition by Miles Reid. Springer-Verlag, Berlin, 1994.

[10] G. Williamson, *An Introduction to Equivariant Sheaves*, available at www.math.uni-freiburg.de/ $\sim$ geordie/.