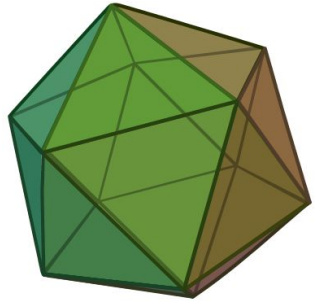
Challenges in the representation theory of finite groups

Geordie Williamson
Max Planck Institute, Bonn



Düsseldorf colloquium in mathematics,
July 2016.

First steps in representation theory

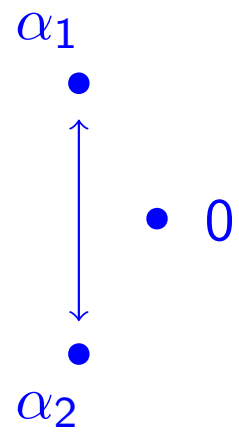
We owe the term *group(e)* to Galois (1832).

Caveat: Actually this might not be true. It is possible that the term occurs earlier in Ruffini (1799).

Galois theory:

$f \in \mathbb{Q}[x]$ $\qquad$ $x^2 + x + 1 = \frac{x^3-1}{x-1}$

$\{\alpha_i\}$ roots of f



Form $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ $\qquad$ $\mathbb{Q}(e^{2\pi i/3})$.

$$\mathrm{Gal}(K, \mathbb{Q}) := \mathrm{Aut}(\mathbb{Q}(\alpha_1, \ldots, \alpha_m)) \text{ ("Galois group")}$$

$$\mathrm{Gal}(K, \mathbb{Q}) \text{ acts on } \{\alpha_1, \ldots, \alpha_m\}.$$

Galois theory: This action tells us everything about f and its roots.

En d'autres termes, quand un groupe G en contient un autre H, le groupe G peut se partager en groupes, que l'on obtient chacun en opérant sur les permutations de H une même substitution ; en sorte que

$$G = H + HS + HS' + \dots .$$

1. Écrite la veille de la mort de l'auteur. (Insérée en 1832 dans la *Revue encyclopédique*, numéro de septembre, page 568.) (J. LIOUVILLE.)

$H \subset G$ is a subgroup

Letter to Auguste Chevalier in 1832

written on the eve of Galois' death

Et aussi il peut se diviser en groupes qui ont tous les mêmes substitutions, en sorte que

$$G = H + TH + T'H + \dots .$$

Ces deux genres de décompositions ne coïncident pas ordinairement. Quand ils coïncident, la décomposition est dite *propre*.

notion of a normal subgroup

Il est aisé de voir que, quand le groupe d'une équation n'est susceptible d'aucune décomposition propre, on aura beau transformer cette équation, les groupes des équations transformées auront toujours le même nombre de permutations.

notion of a simple group

Au contraire, quand le groupe d'une équation est susceptible d'une décomposition propre, en sorte qu'il se partage en M groupes de N permutations, on pourra résoudre l'équation donnée au moyen de deux équations : l'une aura un groupe de M permutations, l'autre un de N permutations.

Lors donc qu'on aura épuisé sur le groupe d'une équation tout ce qu'il y a de décompositions propres possibles sur ce groupe, on arrivera à des groupes qu'on pourra transformer, mais dont les permutations seront toujours en même nombre.

notion of a soluble group

Si ces groupes ont chacun un nombre premier de permutations, l'équation sera soluble par radicaux ; sinon, non.

main theorem of Galois theory

Representation theory is the study of linear group actions:

A *representation* of a group G is a homomorphism

$$\rho : G \to GL(V)$$

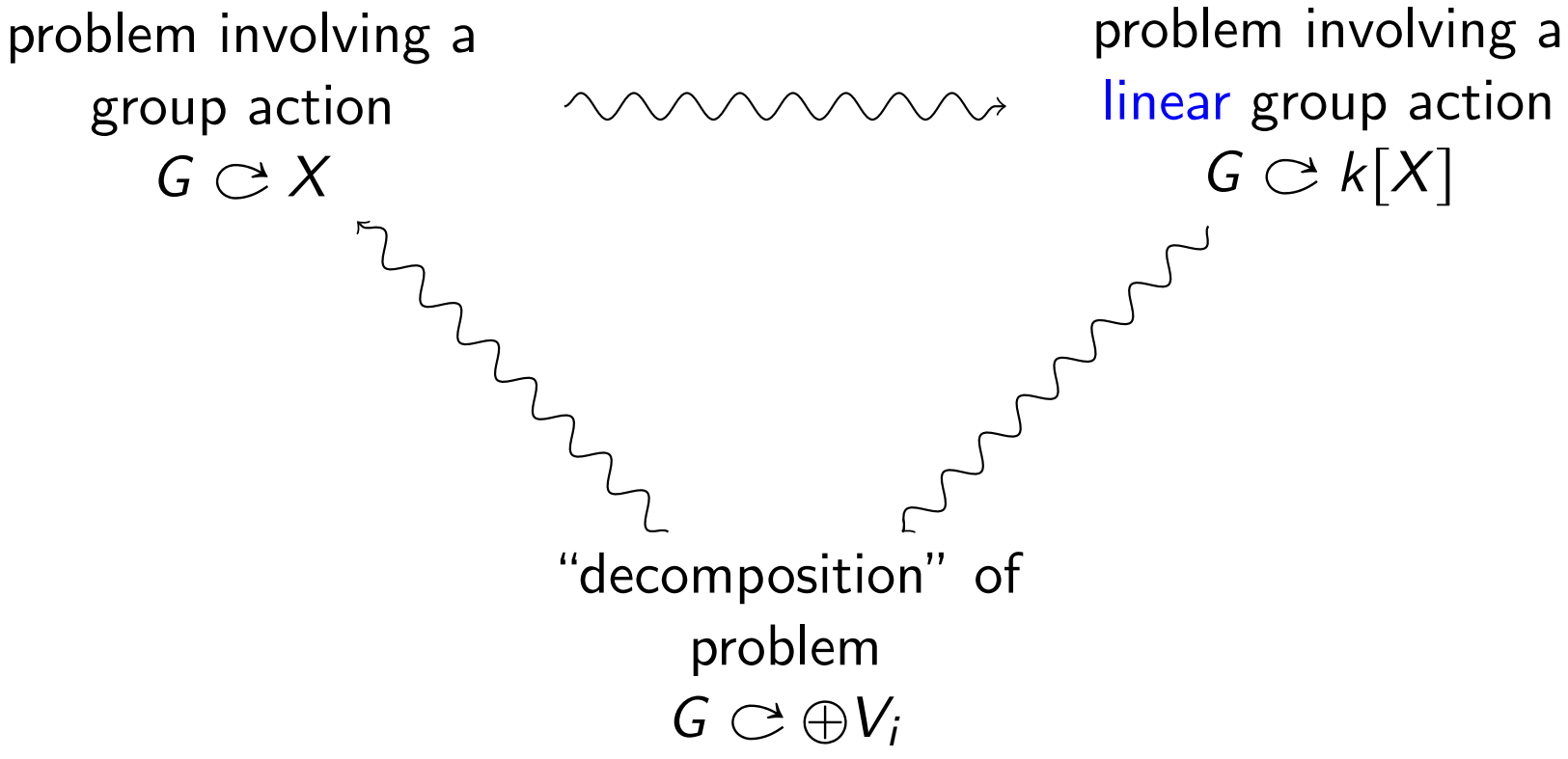
for some vector space V.

A representation is the same thing as a linear action of G on V.

A representation is *irreducible* if the only subspaces $U \subset V$ which are stable under the action of G are $\{0\} \subset V$ and V itself.

There is a Jordan-Hölder theorem: the irreducible representations are the building blocks of all representations.

A representation theorist's strategy:



Three examples of mathematics in light of representation theory

Example 1: Finite group actions on sets.

For a fixed finite group G these two problems are "the same":
1) classify finite sets with G-action;
2) classify subgroups $H \subset G$ up to conjugacy.

The equivalent problems turn out to be extremely complicated. Because every finite group is a subgroup of a symmetric group, a solution to (2) would be something like a classification of all finite groups. There are more than 30 papers on the classification of maximal subgroups of the monster simple group.

However the analogous linear problem "classify $\mathbb{C}$-vector spaces with linear G-action" is representation theory. Here we have a satisfactory answer for many groups.

Example 2: The circle and the Fourier transform.

Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Then S^1 is a (Lie) group.

For any $m \in \mathbb{Z}$ we have a one-dimensional representation of S^1 via:

$$S^1 \in z \mapsto z^m \in \mathbb{C}^* = GL_1(\mathbb{C}).$$

In fact, these are all irreducible representations of S^1!

Now we consider: $S^1 \circlearrowright S^1$.

We linearize this action and consider for example

$$S^1 \circlearrowright L^2(S^1, \mathbb{C}).$$

Now our irreducible characters z^m belong to the right hand side.

Moreover, as Hilbert spaces:

$$L^2(S^1, \mathbb{C}) = \hat{\bigoplus} \mathbb{C} z^m$$

If we identify $S^1 = \mathbb{R}/\mathbb{Z}$ then the functions z^m become the fundamental frequencies $\lambda \mapsto e^{2\pi i m \lambda}$ of Fourier analysis.

Moral: The decomposition of $L^2(S^1, \mathbb{C})$ into irreducible representations is the theory of Fourier series.

Similarly, the Fourier transform can be explained in terms of representations of $(\mathbb{R}, +)$, spherical harmonics in terms of representations of $SO(3) \circlearrowright S^2$, ...

Example 3: Rational points and Fermat's last theorem.

Suppose we want to find rational solutions to an equation X like:

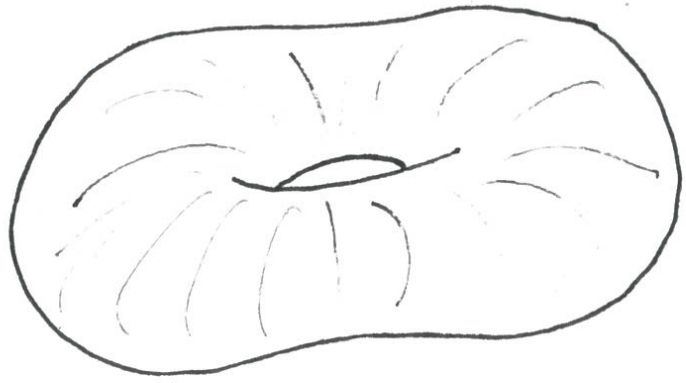
$$y^2 = x^3 - x^2 - 24649x + 1355209$$

Let us write $X(\mathbb{C})$ for the solutions with $x, y \in \mathbb{C}$, $X(\mathbb{Q})$ for solutions $x, y \in \mathbb{Q}$ etc.

It turns out that $X(\mathbb{C})$ is a Riemann surface of genus one:

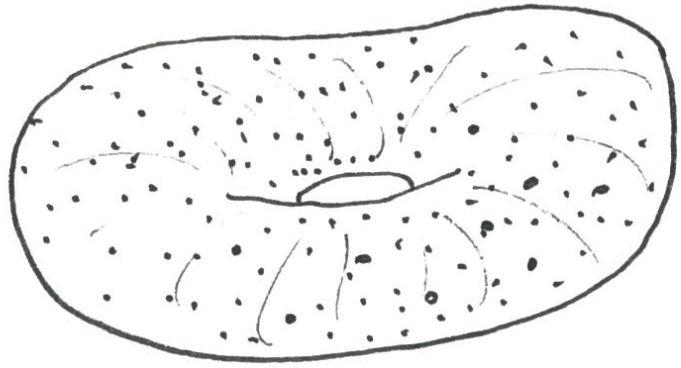
The points in an algebraic closure $X(\overline{\mathbb{Q}})$ are also "easy" (think of the stars in the night sky):

$X(\mathbb{C}) =$

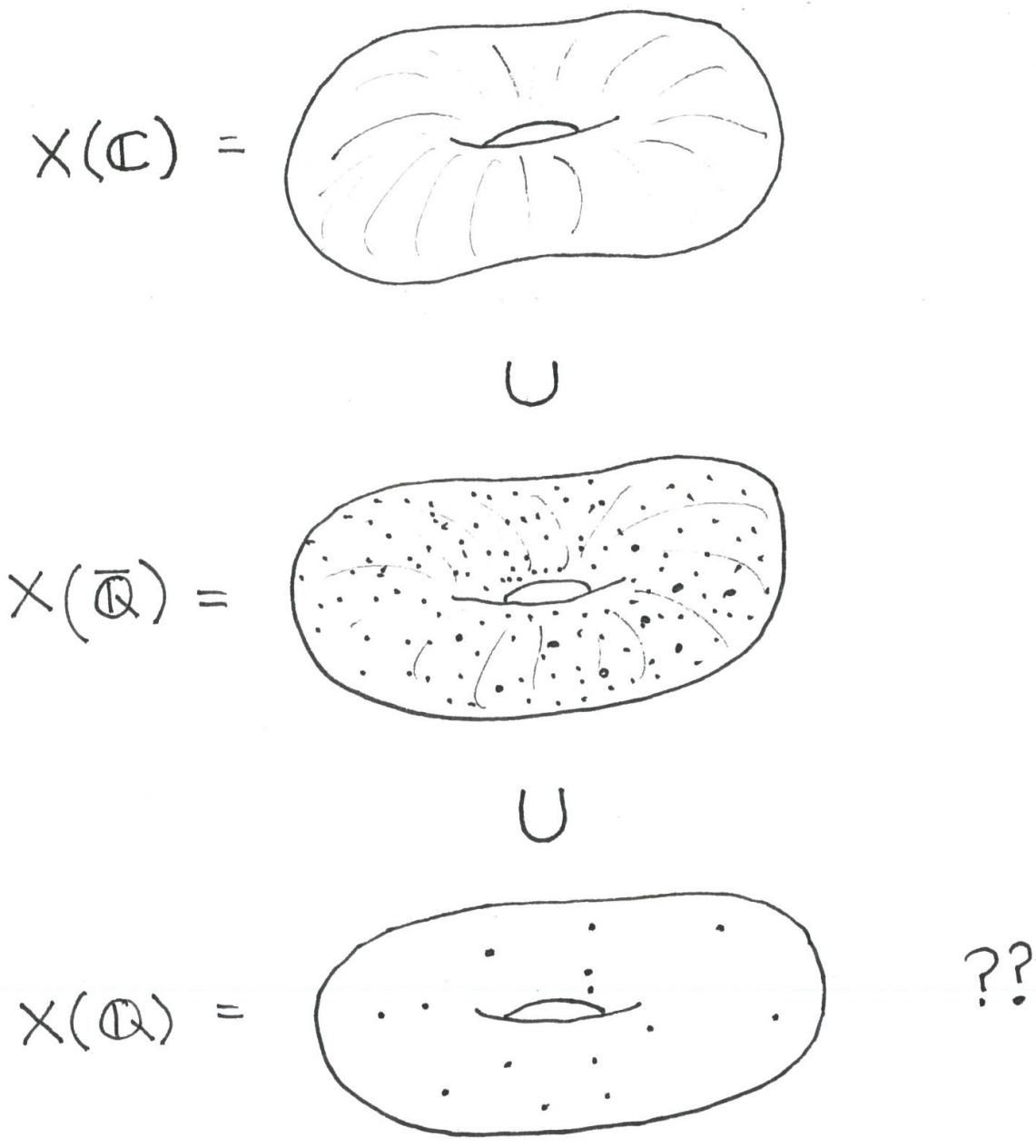


$\cup$

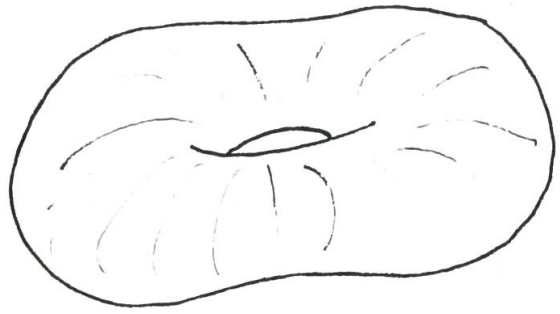
$X(\overline{\mathbb{Q}}) =$



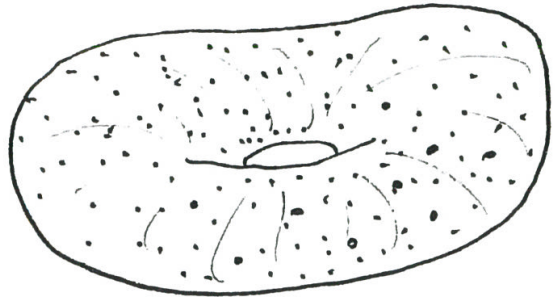
The tricky point is to find the rational points $X(\mathbb{Q})$:



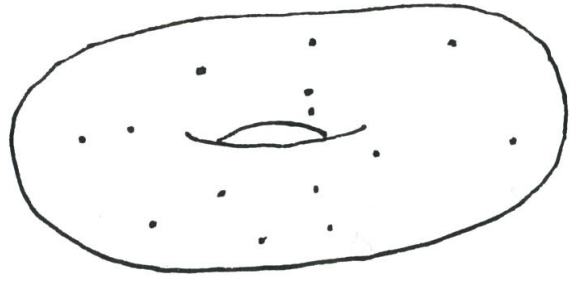
Let $\mathrm{Gal}(\mathbb{Q})$ denote the absolute Galois group (automorphisms of $\mathbb{Q} \subset \overline{\mathbb{Q}}$). Group theory interpretation:

$X(\mathbb{C}) =$ 

$\cup$

$X(\overline{\mathbb{Q}}) =$  $\circlearrowleft$ $\mathrm{Gal}(\overline{\mathbb{Q}})$

$\cup$

$X(\mathbb{Q}) =$  $= X(\overline{\mathbb{Q}})^{\mathrm{Gal}(\overline{\mathbb{Q}})}$

"fixed points"

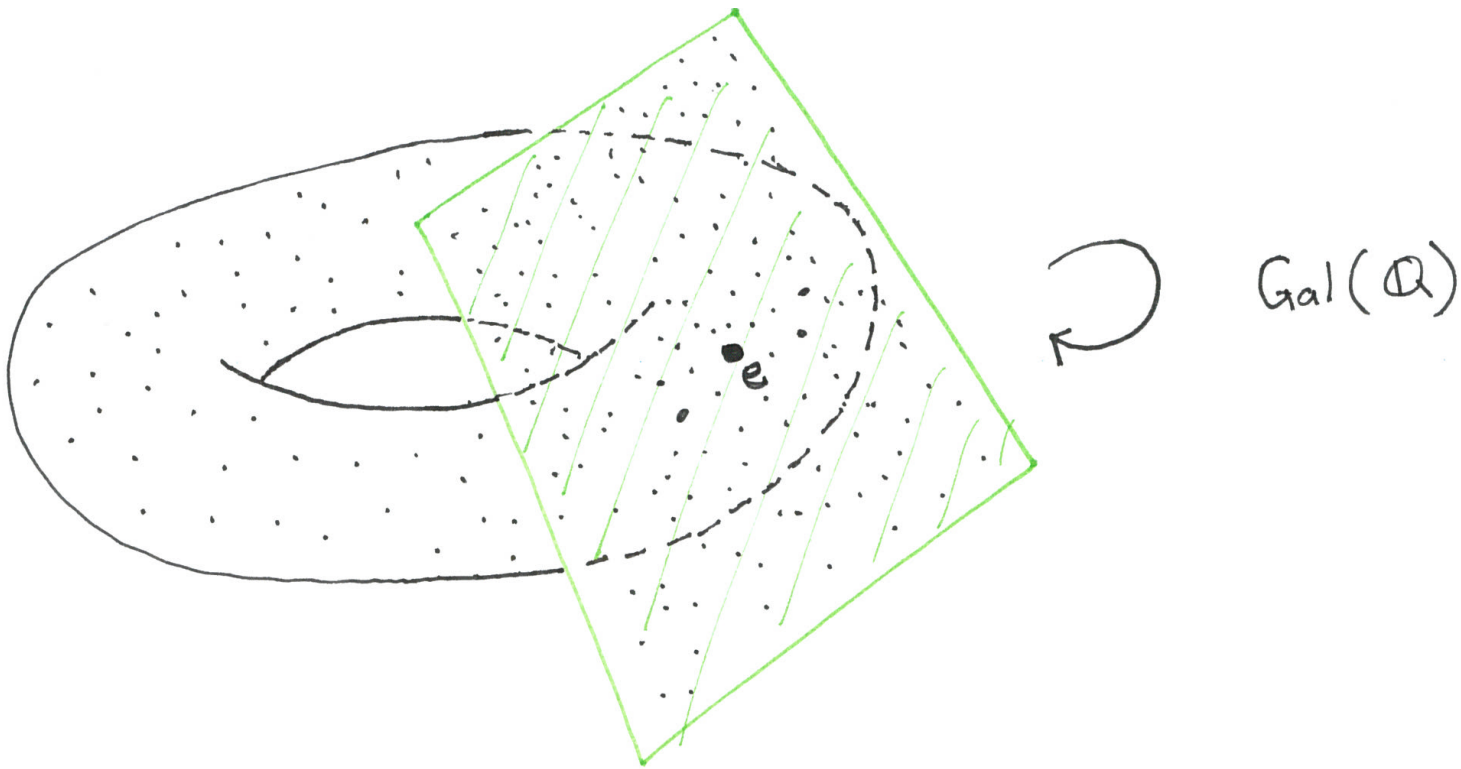
Diophantine geometry can be encoded in questions like:

Understand the $\mathrm{Gal}(\mathbb{Q})$-action on $X(\overline{\mathbb{Q}})$.

But we will probably never understand the $\mathrm{Gal}(\mathbb{Q})$ sets $X(\overline{\mathbb{Q}})$.

However representation theory suggests that we should cook up a linear object out of the action of $\mathrm{Gal}(\mathbb{Q})$ out of $X(\overline{\mathbb{Q}})$.

It turns out that we can do this, and it is *extremely* profitable. The short version: $\mathrm{Gal}(\mathbb{Q})$ acts in a very interesting way on $H_1(X;\mathbb{Q}_\ell) = \mathbb{Q}_\ell^2$. (Can be thought of as something like a tangent space.)



This is the structure behind the proof of Fermat's last theorem:

1. start with a solution $x^n + y^n = z^n$ with $x, y, z \in \mathbb{Z}$, $n > 2$;
2. build from this solution a strange elliptic curve E (the "Frey curve");
3. observe that such a curve would give a very strange G-representation $H_1(E; \mathbb{Q}_3)$ (Frey, Serre, Ribet);
4. show that such a G-representation cannot exist (Wiles, Taylor-Wiles).

Moreover the Langlands program gives us a vast array of theorems and conjectures linking representations of Galois groups coming from Diophantine problems (like the rational points question above) to analysis and automorphic forms. The bridge between these two worlds is provided by representation theory.

A beautiful introduction to these ideas:

R. P. Langlands, *Representation theory: its rise and its role in number theory.* Proceedings of the Gibbs Symposium (New Haven, CT, 1989)

Representations of finite groups and the character table

Basic theorems in the representation theory of a finite group G:

1. any $\mathbb{C}$-representation of G is isomorphic to a direct sum of irreducible representations ("semi-simplicity");
2. $$\#\left\{\begin{array}{c}\text{irreducible}\\ \mathbb{C}\text{-representations of } G\end{array}\right\}_{/\cong} = \#\left\{\begin{array}{c}\text{conjugacy}\\ \text{classes in } G\end{array}\right\}.$$
3. Any finite dimensional representation $\rho : G \to GL(V)$ is determined (up to isomorphism) by its *character*:
$$\chi_\rho : G \to \mathbb{C} : g \mapsto \operatorname{Tr} \rho(g).$$

Hence, we know (almost) everything about the $\mathbb{C}$-representations of a group once we know the characters of the irreducible representations of our group G.

$$\chi(hgh^{-1}) = \mathsf{Tr}(\rho(hgh^{-1})) = \mathsf{Tr}(\rho(h)\rho(g)\rho(h)^{-1}) = \mathsf{Tr}(\rho(g)) = \chi(g).$$

Hence χ is a function on the conjugacy classes of G.

All of this information can be conveniently displayed in the *character table* of G. The rows give the irreducible characters of G and the columns are indexed by the conjugacy classes of G.

The character table of G is the $\mathbb{C}$-linear shadow of G.

The first character table ever published. Here G is the alternating group on 4 letters, or equivalently the symmetries of the tetrahedron.

dnung 3 zwei inverse Classen (2) und $(3) = (2')$. Sei ρ eine prim
bische Wurzel der Einheit.

Tetraeder. $h = 12$.

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	h_α
χ_0	1	3	1	1	1
χ_1	1	-1	1	1	3
χ_2	1	0	ρ	ρ^2	4
χ_3	1	0	ρ^2	ρ	4

Die Werthe von χ_0 sind zugleich die von $f = e$.

Frobenius, *Über Gruppencharaktere*, S'ber. Akad. Wiss. Berlin, 1896.

Now $G = S_5$, the symmetric group on 5 letters of order 120:

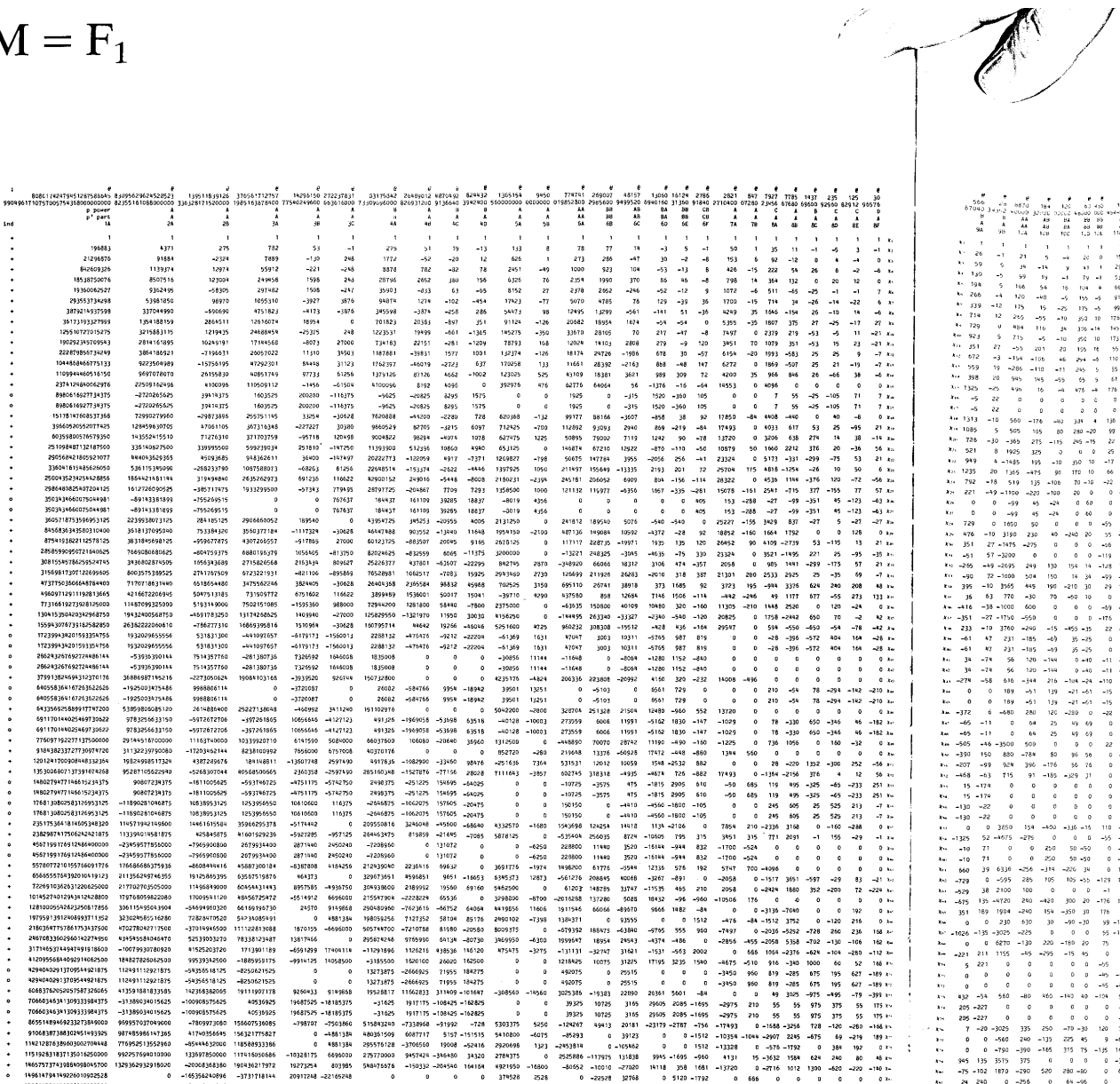
$h = 120$

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	$\chi^{(4)}$	$\chi^{(5)}$	$\chi^{(6)}$	h_α
χ_0	1	5	5	4	4	6	1	1
χ_1	1	1	1	0	0	−2	1	15
χ_2	1	1	−1	2	−2	0	−1	10
χ_3	1	−1	−1	1	1	0	1	20
χ_4	1	−1	1	0	0	0	−1	30
χ_5	1	0	0	−1	−1	1	1	24
χ_6	1	1	−1	−1	1	0	−1	20

1. Frobenius discovered the character table via a different route. He was led to consider this problem following a letter from Dedekind, who related a problem that had interested him *fourteen years* earlier!
2. His first paper also contains the character table of $SL_2(\mathbb{F}_p)$. A few years later he determined the character table of the symmetric group, which has gone on to become universally useful (in pure and applied mathematics, physics, computer science, chemistry, biology ...). Both results remain non-trivial today.

Conway, Curtis, Norton, Parker, Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With computational assistance from J. G. Thackray.* Oxford University Press, 1985.

$$\mathrm{M} = \mathrm{F}_1$$



$$M = F_1$$

	;	e	e	e	e	
		808017424794512875886 45	8309562962452852 3	139511839126	37656171275 7	1429615
		9904961710757005754368000000000	8235516108800000 0	336328171520000	19851638784 00	7754024960
		p power	A	A	A	
		p' part	A	A	A	
	Ind		1A	2A	2B	3A
x_1	+		1	1	1	1
x_2	+		196883	4371	275	782
x_3	+		21296876	91884	-2324	7889
x_4	+		842609326	1139374	12974	55912
x_5	+		18538750076	8507516	123004	249458
x_6	+		19360062527	9362495	-58305	297482
x_7	+		293553734298	53981850	98970	1055310
x_8	+		3879214937598	337044990	-690690	4751823

However around 1900 other mathematicians took some convincing at to the utility of representation theory...

Cayley's dictum that "a group is defined by means of the laws of combination of its symbols" would imply that, in dealing purely with the theory of groups, no more concrete mode of representation should be used than is absolutely necessary. It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

– Burnside, *Theory of groups of finite order*, 1897.

(One year after Frobenius' definition of the character.)

PREFACE TO THE SECOND EDITION

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is accordingly in the present edition a large amount of new matter.

– Burnside, *Theory of groups of finite order*, Second edition, 1911.
(15 years after Frobenius' definition of the character table.)

First steps in modular representation theory

We have so far discussed representations over $\mathbb{C}$.

The story remains the same over fields of characteristic not dividing $|G|$.

However over fields of small characteristic the situations becomes much more complicated.

In fact, any representation of G over a field of characteristic p is completely reducible if and only if p does not divide $|G|$.

Why study modular representations?

1. Provides a way of recognising groups. (If I suspect that $G \cong SL_n(\mathbb{F}_q)$, I might like to proceed by constructing a representation of G on $\mathbb{F}_q^n$.)
2. Explains deep properties of the reduction modulo p of the character table.
3. Many representations occurring in (mathematical) nature are modular representations. (In number theory, algebraic geometry, ...)
4. If a high power of p divides the order of G then the category of representations of G is extremely complicated. It is possible that this explains that recent interest in the subject (a source of "small" abelian categories with highly intricate structure).

Modular representation theory was initially developed almost single handedly by Richard Brauer (1901 - 1977) from 1935 - 1960.

Brauer's interest in representation theory seems have been motivated by a lifelong interest in number theory, as well as an fascination for the structure of finite groups. Brauer's results are widely regarded as providing the first steps towards the classification of finite simple groups.

The most useful technique for studying modular representations is "reduction modulo p".

Take an irreducible representation of G, say over $\mathbb{Q}$:

$$\rho : G \to GL_n(\mathbb{Q}).$$

It can always be realised by integral matrices:

$$\rho : G \to GL_n(\mathbb{Z}).$$

Reducing it modulo p gives a representation

$$\overline{\rho} : G \to GL_n(\mathbb{F}_p).$$

Decomposing $\overline{\rho}$ into irreducible pieces (determining the "decomposition numbers") can be an extremely difficult problem.

($\overline{\rho}$ depends on choices, but its irreducible pieces ("class in the Grothendieck group") does not.)

An example:

Both representations $(+1)$ and (-1) of $G = \mathbb{Z}/2\mathbb{Z}$ become isomorphic if we reduce modulo 2.

They stay distinct for all other primes.

Another example: Consider the representation of the symmetric group S_n via permutations on

$$V := \{v = (v_1, \ldots, v_n) \in \mathbb{Q}^n \mid \sum v_i = 0\}.$$

It has a natural integral form:

$$V_{\mathbb{Z}} := \{v = (v_1, \ldots, v_n) \in \mathbb{Z}^n \mid \sum v_i = 0\}.$$

Reducing modulo p we obtain:

$$V_{\mathbb{F}_p}\{v = (v_1, \ldots, v_n) \in \mathbb{F}_p^n \mid \sum v_i = 0\}.$$

If p divides n then $V_{\mathbb{F}_p}$ contains the trivial submodule:

$$\{(\lambda, \ldots, \lambda) \mid \lambda \in \mathbb{F}_p\} \subset V_{\mathbb{F}_p}. \quad \text{(because } \sum_{i=1}^{n} \lambda = n\lambda = 0!\text{)}$$

In fact, $V_{\mathbb{F}_p}$ is irreducible if and only if p does not divide n.

A more complicated example:

If we reduce the 196883 dimensional irreducible representation of the monster simple group modulo 2 we obtain

irreducible mod 2 rep + trivial representation.

(This 196882 dimensional representation over $\mathbb{F}_2$ is the most efficient representation of the monster known.)

Update 15th December 1998: standard generators have now been made as 196882 $\times$ 196882 matrices over $\mathbb{F}_2$ this took about 8 hours CPU time on a pentium machine. They have been multiplied together, using most of the computing resources of Lehrstuhl D fr̈ Mathematik, RWTH Aachen, for about 45 hours completed 05:40 on December 14th.

From Rob Wilson's Atlas site: http://brauer.maths.qmul.ac.uk/Atlas/spor/M/

The Lusztig conjecture and the James conjecture

We will concentrate on the following basic questions:

Describe the irreducible modular representations of the symmetric group S_n.

Describe the irreducible modular representations of a (split) finite group of Lie type (e.g. $GL_n(\mathbb{F}_q)$, $Sp_{2n}(\mathbb{F}_q)$, $E_8(\mathbb{F}_q)$ ...) in *natural* characteristic (i.e. in characteristic p where $q = p^r$).

By the classification of finite simple groups, all but 26 exceptional "sporadic" simple groups are close relatives of the above groups.

The basic structure of our knowledge in both cases is the same: it is not difficult to write down a set parametrising the irreducible representations, however the structure of the categories is extremely complicated and largely unknown.

Examples:

1. (Rouquier, Bridgeland) There are known or conjectured derived equivalences in modular representation theory which are close relatives of derived equivalences occurring in the birational geometry of algebraic varieties ("crepant resolution conjecture"). This is a modern version of the question "can two groups have isomorphic character tables?".
2. (Arkhipov-Bezrukavnikov-Ginzburg) Generic versions of these categories (over $\mathbb{C}$) occur as basic ingredients in the tamely ramified geometric Langlands equivalence.

It is remarkable observation (Verma, Jantzen, Andersen, Lusztig, James, ...) that in both the above cases (modular representations of symmetric groups, and natural characteristic representations of finite groups of Lie type) the decomposition numbers appear to become "uniform in p" as soon as p is "not too small".

Example: For $n \leqslant 6$, the (highly non semi-simple) representation theory of $GL_n(\mathbb{F}_p)$ in characteristic p appears to become uniform as long as $p \geqslant n$. (If $p < n$ things appear much harder ...)

This conjectural stabilisation for groups of Lie type (e.g. $GL_n(\mathbb{F}_q)$) is made precise by the *Lusztig conjecture*.

The conjectural stabilisation for the symmetric group is made precise by the *James conjecture*.

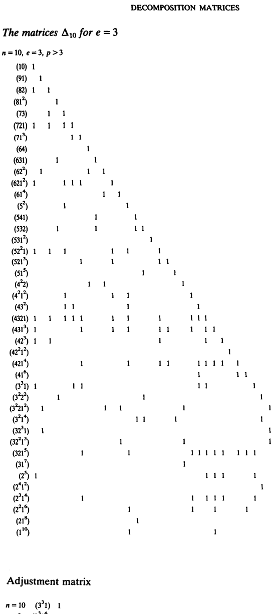
Gordon James formulated his conjecture in 1990 following formidable calculations. He conjectured a formula for the decomposition numbers of simple representations of S_n if $p > \sqrt{n}$ ("p not too small").

Roughly speaking, his conjecture says that the Hecke algebra at a p^{th}-root if $p > \sqrt{n}$ of unity sees all the complexity of mod p representation theory.

His conjecture, if true, would represent major progress on the problem.

His conjecture is true for $n = 1, 2, \ldots, 22$.

James, *The decomposition matrices of $GL_n(q)$ for $n \leqslant 10$*, Proc. London Math. Soc. (3) 60 (1990), no. 2, 225–265.


DECOMPOSITION MATRICES

The matrices Δ_{10} for $e = 3$

$n = 10,\ e = 3,\ p > 3$

Adjustment matrix

$n = 10$ $(3^3 1)$ 1

Lusztig's conjecture (1980).

Proceedings of Symposia in Pure Mathematics
Volume 37, 1980

SOME PROBLEMS IN THE REPRESENTATION THEORY OF FINITE CHEVALLEY GROUPS

GEORGE LUSZTIG[1]

obtained by reducing modulo p the irreducible representation with highest weight $-w\rho-\rho$ of the corresponding complex group. (It is well defined in the Grothendieck group.) We assume that $\alpha_0^\vee(\rho)<p$.

Problem IV. *Assume that w is dominant and it satisfies the Jantzen condition $\alpha_0^\vee(-w\rho)\leqslant p(p-h+2)$, where h is the Coxeter number. Then*

$$\text{ch } L_w=\sum_{\substack{y\in W_a,\ \text{dominant}\\ y\leqslant w}}(-1)^{l(w)-l(y)}P_{y,w}(1)\text{ch } V_y. \tag{4}$$

From this, one can deduce the character formula for any irreducible finite dimensional representation of G (over $\bar{\mathbf{F}}_p$), by making use of results of Jantzen and Steinberg. The evidence for this character formula is very strong. I have verified it in the cases where G is of type A_2, B_2 or G_2. (In these cases, ch L_w has been computed by Jantzen.) One can show using results of Jantzen [**2**, Anhang]

What bound (e.g. for which p Lusztig's formula holds for $GL_n(\mathbb{F}_p)$) on the previous slide has been one of the central puzzles in modular representation theory over the last thirty years.

What "large" means on the previous slide is a tricky business.

Let h denote the Coxeter number of G

(e.g. $h = n$ for GL_n, $h = 2n$ for SP_{2n}, $h = 30$ for E_8):

1. 1980: Lusztig conjectured $p \geqslant 2h - 3$ (Jantzen condition);
2. 1985: Kato conjectured $p \geqslant h$;
3. 1994: Several hundred pages of Andersen-Jantzen-Soergel, Kazhdan-Lusztig, Kashiwara-Tanisaki and Lusztig prove the conjecture for large p without any explicit bound!

W. Soergel (2000): "Bei Wurzelsystemen verschieden von A_2, B_2, G_2, A_3, weiß man aber für keine einzige Charakteristik ob sie hinreichend groß ist."
. . . a particularly strange situation for finite group theorists.

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2. 1985: Kato conjectured $p \geqslant h$;
3. 1994: Andersen-Jantzen-Soergel, Kazhdan-Lusztig, Kashiwara-Tanisaki, Lusztig: the conjecture holds for large p;
4. 2008: Fiebig gave an explicit enormous bound (e.g. $p > 10^{40}$ for $SL_9(\mathbb{F}_p)$ against the hoped for $p \geqslant 11$)!

The following 2013 theorem has a part joint with Xuhua He and another part joint with Alex Kontorovich and Peter McNamara, and builds on work done in a long term project with Ben Elias.

Theorem

There exists a constants $a > 0$ and $c > 1$ such that Lusztig's conjecture on representations of $SL_n(\mathbb{F}_p)$ fails for many primes $p > ac^n$ and $n \gg 0$.

The theorem implies that there is no polynomial bound in the Coxeter number for the validity of Lusztig's conjecture. This should be compared with the hope (believed by many for over thirty years) that the bound is a simple linear function of Coxeter number.

Provably we can take $a = 5/7$ and $c = 1.101$. Experimentally c can be taken much larger. For example, Lusztig's conjecture fails for $SL_{100}(\mathbb{F}_p)$ with $p = 470\ 858\ 183$.

It is disconcerting (or enlivening?) that there is some interesting number theory behind the above growth rates.

Following a line of attack suggested by Joe Chuang, the previous result also yields:

Theorem

The James conjecture fails "generically". In particular, it is not true for S_n for all $n \geqslant 1\,744\,860$.

The proof proceeds by constructing certain representations that are (much) smaller than the James conjecture predicts.

Following a line of attack suggested by Joe Chuang, the previous result also yields:

Theorem

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Another key tool are techniques going back to Schur's PhD thesis in Berlin in 1901 (one year after Frobenius first wrote down the character table of the symmetric group)!

We are trying to work out where, between $n = 22$ and $n = 1\,744\,860$, the conjecture first goes wrong.

There is still much to say about S_n, possibly the most fundamental of all finite groups ...

A key step in establishing this theorem is a "translation of the problem into topology" completed by Wolfgang Soergel in 2000.

This is an instance of "geometric representation theory": the topology of complex algebraic varieties has much to say about representation theory.

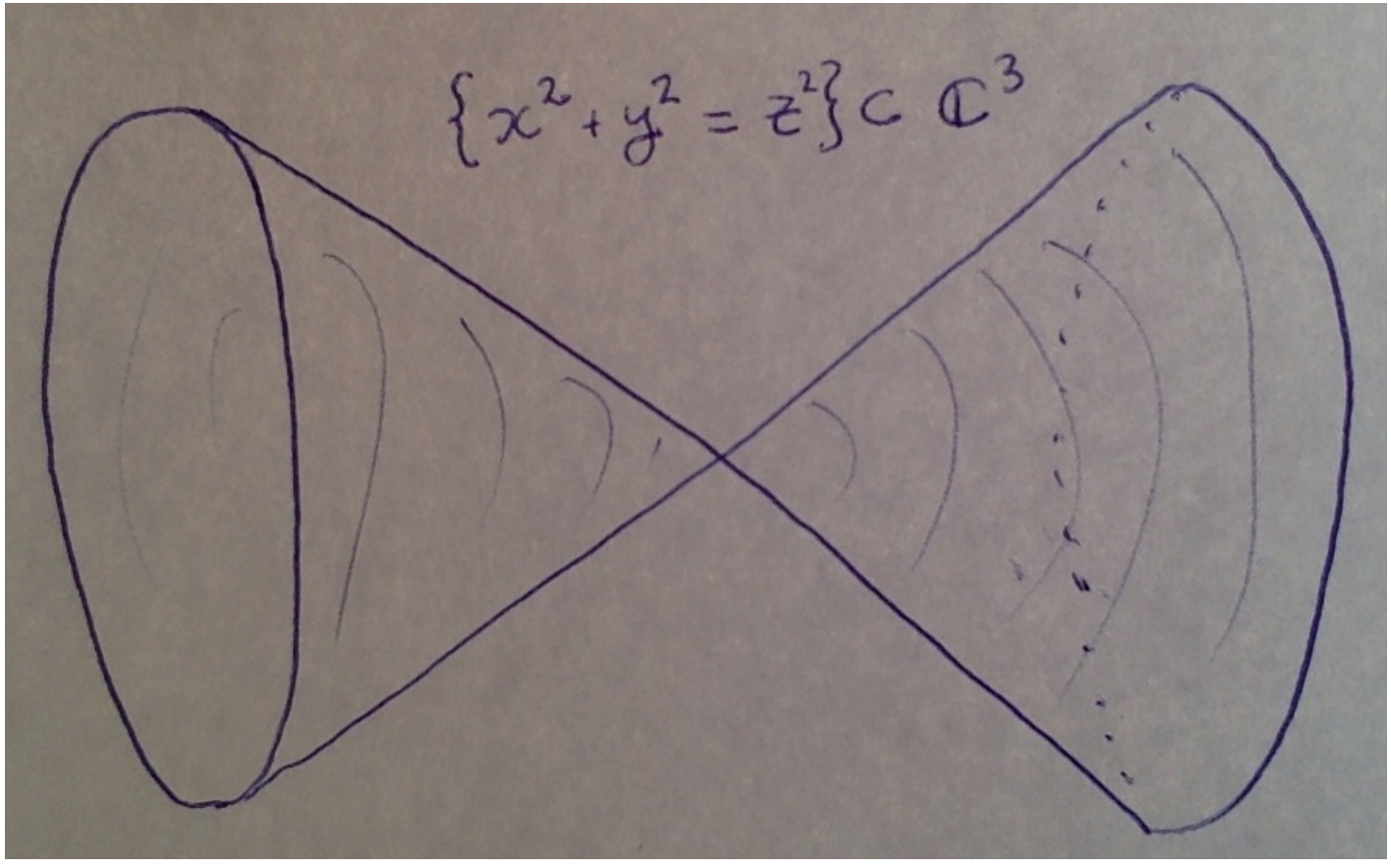
This field has been driven by Lusztig and many others over the past forty years. It must sadly stay a black box in this talk.

Example: The characters of $GL_n(\mathbb{F}_q)$ may be described via certain geometric objects ("character sheaves") which live on of $GL_n(\mathbb{C})$. Thus there is a geometric procedure to produce the character table of $GL_n(\mathbb{F}_q)$ for "all q's at once".

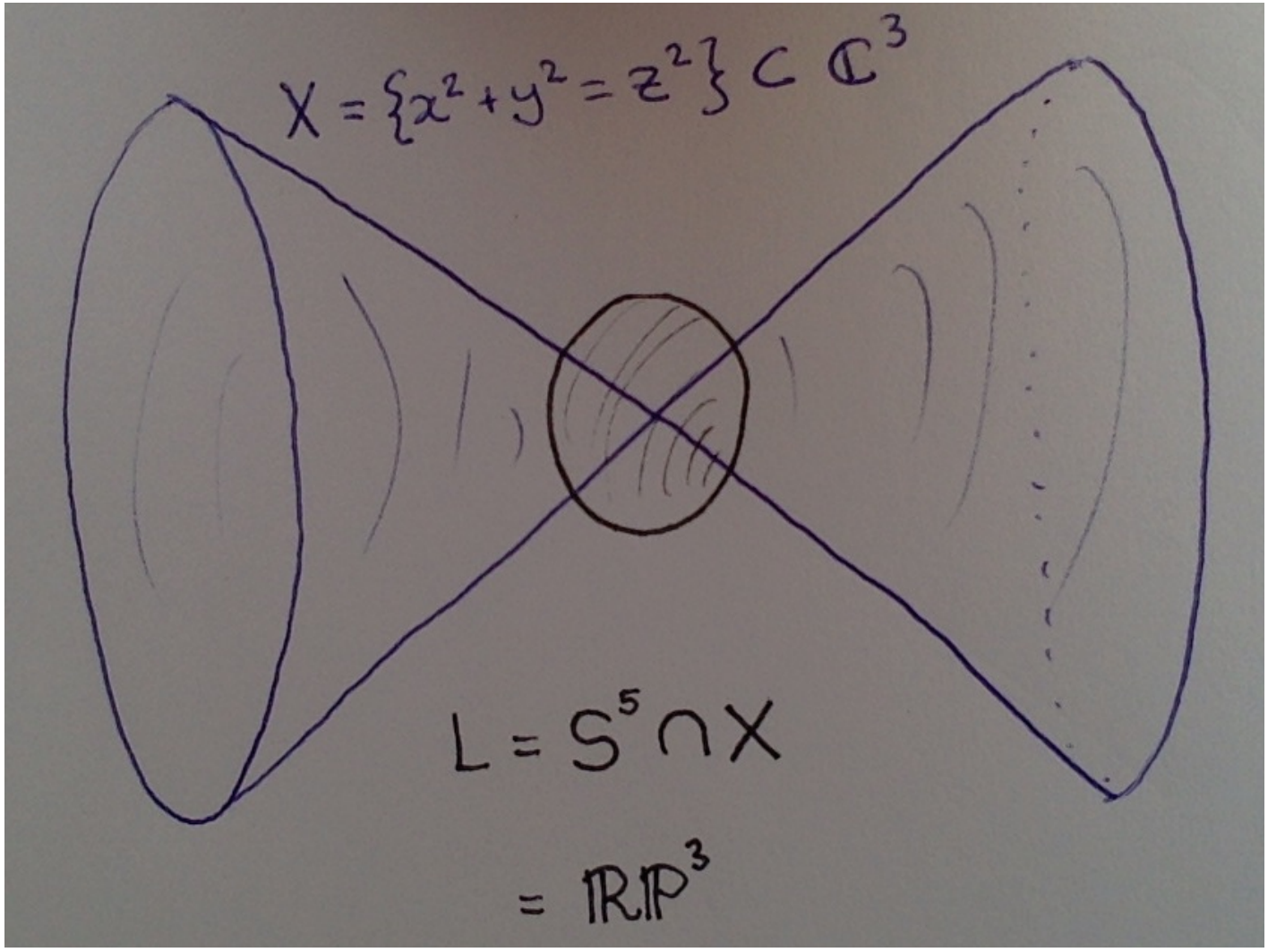
Roughly speaking, the coefficients where one takes representations corresponds to the coefficients of cohomology.

Example:

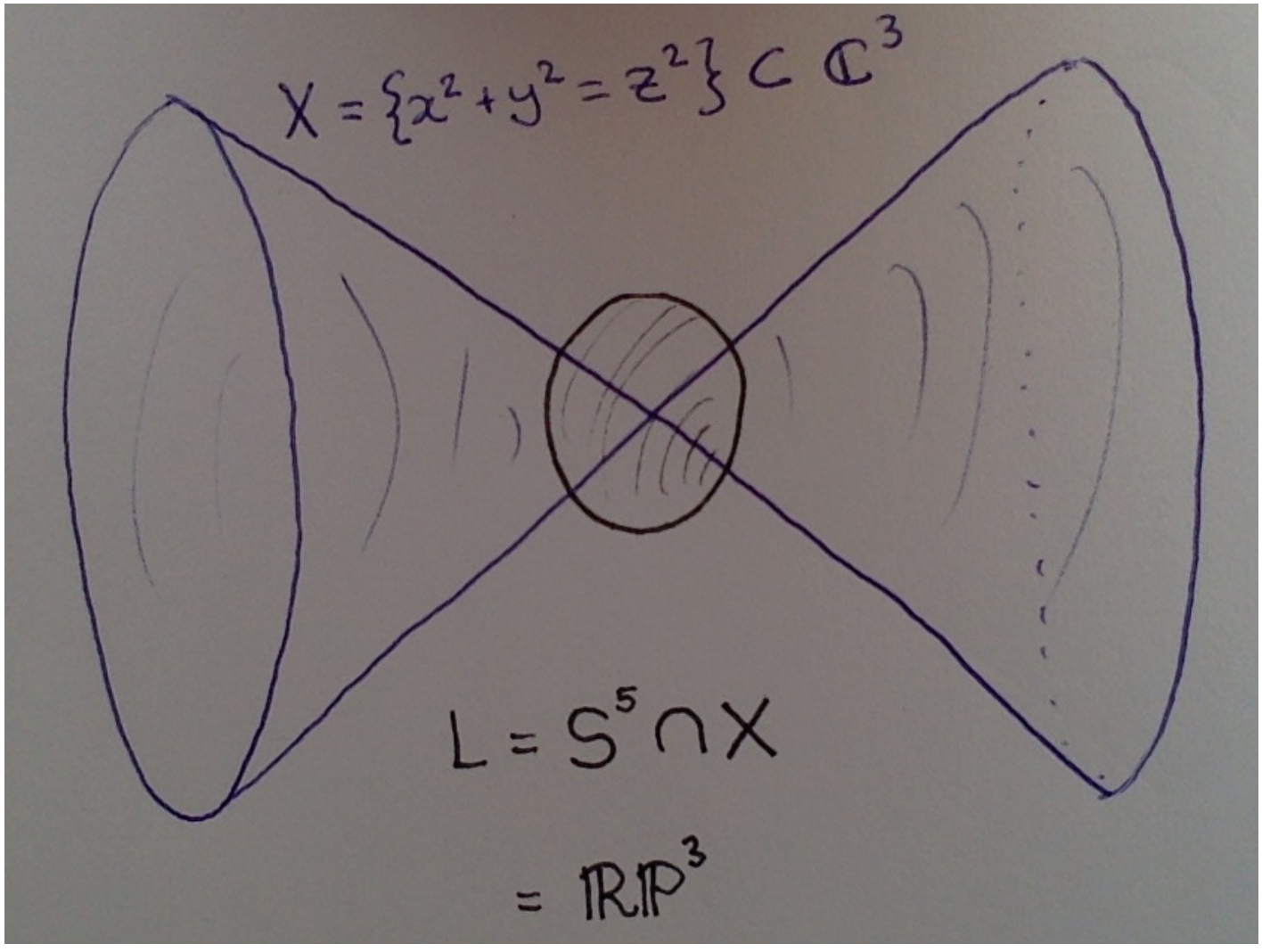
Consider the quadric cone ($\dim_{\mathbb{C}} = 2$, singular space). We can draw a real picture (remember that a lot is missing!):



If we intersect a small sphere around the singularity with X we obtain ...



Hint: $X = \mathbb{C}^2/(\pm 1)$ so $L = S^3/\pm 1 = \mathbb{RP}^3$.



The we have $H^2(\mathbb{RP}^3) = \mathbb{Z}/2\mathbb{Z}$ and all other groups are torsion free. This turns out to be *equivalent* to the fact that the representation theory of $\mathbb{Z}/2\mathbb{Z}$ is "different" in characterstic 2.

Theorem

Let c be a non-zero coefficient of a word w of length ℓ in the generators:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then associated to w one can find $\mathbb{Z}/c\mathbb{Z}$-torsion in a variety controlling the representation theory of $SL_{3\ell+5}$.

In particular, any prime p dividing c which is larger than $3\ell + 5$ gives a counter-example to the expected bounds in Lusztig's conjecture.

Some non-trivial number theory (relying on ideas surrounding the affine sieve and Zaremba's conjecture) yields that the prime divisors of c above grow like $O(c^n)$ for come $c > 1$.

In summary:

The Lusztig and James conjecture predict a remarkable regularity in the modular representation theory of symmetric groups and finite groups of Lie type for large primes.

However it takes much longer for this regularity to show itself than was expected.

For "mid range primes" (e.g. $n < p < c^n$) subtle and unexpected arithmetic questions show up in the representation theory of groups like $GL_n(\mathbb{F}_p)$.

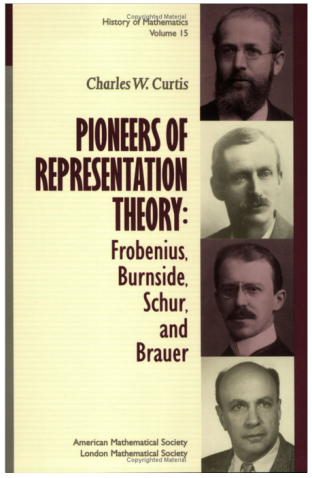
In recent joint work with Simon Riche we have proposed a new conjecture which conjecturally gives an answer for all primes. However we still can't decide exactly where the uniformity of the Lusztig and James conjecture takes over.

However in spite of all our efforts, we know very little about finite groups. The mystery has not been resolved, we cannot even say for sure whether order or chaos reigns. If any excitement can be derived from what I have to say, it should come from the feeling of being at a frontier across which we can see many landmarks, but which as a whole is unexplored, of planning ways to find out about the unknown, even if the pieces we can put together are few and far apart. My hope then is that some of you may go out with the idea: "Now let me think of something better myself."

– Richard Brauer, *On finite groups and their characters*,

Bull. Amer. Math. Soc. Volume 69, Number 2 (1963), 125-130.

Thanks!



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Curtis, *Pioneers of representation theory: Frobenius, Burnside, Schur, and Brauer.* History of Mathematics, 15. AMS, 1999.

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