An example of higher representation theory

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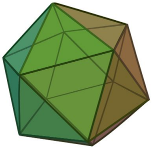


Quantum 2016,
Cordoba, February 2016.

First steps in representation theory.

We owe the term *group(e)* to Galois (1832).

En d'autres termes, quand un groupe G en contient un autre H, le groupe G peut se partager en groupes, que l'on obtient chacun en opérant sur les permutations de H une même substitution ; en sorte que

$$G = H + HS + HS' + \dots.$$

1. Écrite la veille de la mort de l'auteur. (Insérée en 1832 dans la *Revue encyclopédique*, numéro de septembre, page 568.) (J. LIOUVILLE.)

$H \subset G$ is a subgroup

Letter to Auguste Chevalier in 1832

written on the eve of Galois' death

Et aussi il peut se diviser en groupes qui ont tous les mêmes substitutions, en sorte que

$$G = H + TH + T'H + \dots.$$

Ces deux genres de décompositions ne coïncident pas ordinairement. Quand ils coïncident, la décomposition est dite *propre*.

notion of a normal subgroup

Il est aisé de voir que, quand le groupe d'une équation n'est susceptible d'aucune décomposition propre, on aura beau transformer cette équation, les groupes des équations transformées auront toujours le même nombre de permutations.

notion of a simple group

Au contraire, quand le groupe d'une équation est susceptible d'une décomposition propre, en sorte qu'il se partage en M groupes de N permutations, on pourra résoudre l'équation donnée au moyen de deux équations : l'une aura un groupe de M permutations, l'autre un de N permutations.

Lors donc qu'on aura épuisé sur le groupe d'une équation tout ce qu'il y a de décompositions propres possibles sur ce groupe, on arrivera à des groupes qu'on pourra transformer, mais dont les permutations seront toujours en même nombre.

notion of a soluble group

Si ces groupes ont chacun un nombre premier de permutations, l'équation sera soluble par radicaux ; sinon, non.

main theorem of Galois theory

Mathematicians were studying group theory for 60 years before they began studying *representations* of finite groups.

The first character table ever published. Here G is the alternating group on 4 letters, or equivalently the symmetries of the tetrahedron.

dnung 3 zwei inverse Classen (2) und (3) = (2′). Sei ρ eine prim
bische Wurzel der Einheit.

Tetraeder. $h = 12$.

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	h_α
χ_0	1	3	1	1	1
χ_1	1	-1	1	1	3
χ_2	1	0	ρ	ρ^2	4
χ_3	1	0	ρ^2	ρ	4

Die Werthe von χ_0 sind zugleich die von $f = e$.

Frobenius, *Über Gruppencharaktere*, S'ber. Akad. Wiss. Berlin, **1896**.

Now $G = S_5$, the symmetric group on 5 letters of order 120:

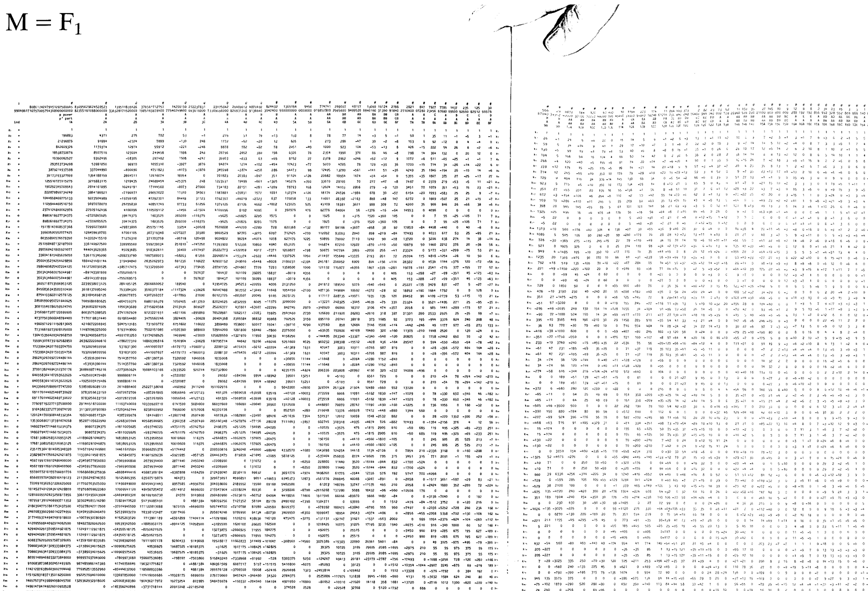
[1013] Frobenius: Über Gruppencharaktere. 29

$h = 120$

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	$\chi^{(4)}$	$\chi^{(5)}$	$\chi^{(6)}$	h_α
χ_0	1	5	5	4	4	6	1	1
χ_1	1	1	1	0	0	−2	1	15
χ_2	1	1	−1	2	−2	0	−1	10
χ_3	1	−1	−1	1	1	0	1	20
χ_4	1	−1	1	0	0	0	−1	30
χ_5	1	0	0	−1	−1	1	1	24
χ_6	1	1	−1	−1	1	0	−1	20

Conway, Curtis, Norton, Parker, Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With computational assistance from J. G. Thackray.* Oxford University Press, 1985.

$\mathrm{M} = \mathrm{F}_1$



However around 1900 other mathematicians took some convincing at to the utility of representation theory...

Cayley's dictum that "a group is defined by means of the laws of combination of its symbols" would imply that, in dealing purely with the theory of groups, no more concrete mode of representation should be used than is absolutely necessary. It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

– Burnside, *Theory of groups of finite order*, 1897.
(One year after Frobenius' definition of the character.)

PREFACE TO THE SECOND EDITION

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is accordingly in the present edition a large amount of new matter.

– Burnside, *Theory of groups of finite order*, Second edition, 1911.
(15 years after Frobenius' definition of the character table.)

Representation theory is useful because symmetry is everywhere and linear algebra is powerful!

Categories can have symmetry too!

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Caution: What “linear” means is more subtle.

Usually it means to study categories in which one has operations like direct sums, limits and colimits, kernels ...

(Using these operations one can try to “categorify linear algebra” by taking sums, cones etc. If we are lucky Ben Elias will have more to say about this.)

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Amusing: Under this analogy difficult conjectures about derived equivalence (e.g. Broué conjecture) are higher categorical versions of questions like "can two groups have the same character table"?

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Partial answer: Groups, Lie algebras, quantum groups, Hecke algebras, . . .

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In higher representation theory we ask:

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I would suggest that we don't know the answer to this question. We are witnessing the birth of a theory. We know some examples which are both intrinsically beautiful and powerful, but are far from a general theory.

R. Rouquier, *2-Kac-Moody algebras*, 2008

Over the past ten years, we have advocated the idea that there should exist monoidal categories (or 2-categories) with an interesting "representation theory": we propose to call "2-representation theory" this higher version of representation theory and to call "2-algebras" those "interesting" monoidal additive categories. The difficulty in pinning down what is a 2-algebra (or a Hopf version) should be compared with the difficulty in defining precisely the meaning of quantum groups (or quantum algebras).

First steps in higher representation theory.

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In higher representation theory we study homomorphisms $\mathcal{A} \to \mathrm{End}(V)$ for $\mathcal{A}$ a monoidal category and ask what such homomorphisms might tell us about $\mathcal{C}$.

Thus algebras are replaced by (additive or sometimes abelian) tensor categories.

Recall: $\mathcal{A}$ is an additive tensor category if we have a bifunctor of additive categories:

$$(M_1, M_2) \mapsto M_1 \otimes M_2$$

together with a unit $\mathbb{1}$, associator, ...

Examples: Vect_k, Rep G, G-graded vector spaces, $\mathrm{End}(\mathcal{C})$ (endofunctors of an additive category), ...

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What exactly this means can take a little getting used to.

As in classical representation theory it is often more useful to think about an "action" of $\mathcal{A}$ on $\mathcal{M}$.

(1) $$\begin{array}{ccc} \mathcal{A} & & \mathcal{M} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (A, & & M) \end{array} \longmapsto A \bullet M$$

"objects act on objects"

(often visible on Grothendieck group)

(2) $$\left(\begin{tikzcd} A \\ A \arrow[u, "f"] \end{tikzcd} , M \right) \longmapsto \begin{tikzcd} A' \bullet M \\ A \bullet M \arrow[u, "f \bullet M"'] \end{tikzcd}$$

"morphisms act on objects"

(3) $$\left(A , \begin{tikzcd} M' \\ M \arrow[u, "g"'] \end{tikzcd} \right) \longmapsto \begin{tikzcd} A \bullet M' \\ A \bullet M \arrow[u, "A \bullet g"'] \end{tikzcd}$$

"objects act on morphisms"

A first example:

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An $\mathcal{A}$-module is a recipe $M \mapsto \mathrm{nat} \cdot M$ and a host of maps

$$\mathsf{Hom}_{\mathcal{A}}(\mathrm{nat}^{\otimes m}, \mathrm{nat}^{\otimes n}) \to \mathsf{Hom}_{\mathcal{M}}(\mathrm{nat}^{\otimes m} \cdot M, \mathrm{nat}^{\otimes n} \cdot M)$$

satisfying an even larger host of identities which I will let you contemplate.

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Theorem

(Classification of representations of $\mathrm{Rep}\, SU_2$*.) These are all.*

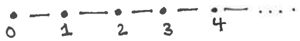
Let $\{L_i\}$ denote the simple objects in $\mathcal{M}$.

Draw an edge $L_i \longrightarrow L_j$ if $L_i \overset{\oplus}{\subset} \mathrm{nat} \cdot L_j$.

<u>Exercise:</u> nat self-dual $\Longrightarrow \left(L_i \longrightarrow L_j \iff L_j \longrightarrow L_i \right)$.

$\mathrm{Vect}_{\mathbb{C}}$

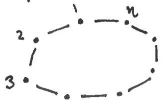
$\mathrm{Rep}\, SU_2$



$\mathrm{Rep}\, BI$

$\mathrm{Rep}\, S^1$ $\left(\mathbb{C}[x, x^{-1}]^{S_2} \subset \mathbb{C}[x, x^{-1}] \right)$

$\mathrm{Rep}\, \mu_n$



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This is an example of "rigidity" in higher representation theory.

An example of higher representation theory
(joint with Simon Riche).

We want to apply these ideas to the modular (i.e. characteristic p) representation theory of finite and algebraic groups.

Here the questions are very difficult and we will probably never know a complete and satisfactory answer.

Some motivation from characteristic 0:

Recall the famous Kazhdan-Lusztig conjecture (1979):

$$\mathrm{ch}(L_w) = \sum_{y \in W} (1)^{\ell(w)-\ell(y)} P_{y,w}(1)\mathrm{ch}(M_y)$$

(Here L_w (resp. M_y) is a simple highest weight module (resp. Verma module) for a complex semi-simple Lie algebra, and $P_{y,w}$ is a "Kazhdan-Lusztig" polynomial.)

The Kazhdan-Lusztig conjecture has 2 distinct proofs:

1. *Geometric:* Apply the localization theorem for $\mathfrak{g}$-modules to pass to differential operators (D-modules) on the flag variety, then pass through the Riemann-Hilbert correspondence to land in perverse sheaves, and using some deep geometric tools (e.g. proof of Weil conjectures) complete the proof (Kazhdan-Lusztig, Beilinson-Bernstein, Brylinsky-Kashiwara 1980s). This proof uses every trick in the book!

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2. *Categorical:* Show that translation functors give an action of "Soergel bimodules" on category $\mathcal{O}$. Then the Kazhdan-Lusztig conjecture follows from the calculation of the character of indecomposable Soergel bimodules (Soergel 1990, Elias-W 2012). This proof is purely algebraic.

We want to apply the *second* approach to the representation theory of reductive algebraic groups.

The first approach has also seen recent progress (Bezrukavnikov-Mirkovic-Rumynin) however it seems much more likely at this stage that the second approach will yield computable character formulas.

For the rest of the talk fix a field k and a connected reductive group G like GL_n (where we will state a theorem later) of Sp_4 (where we can draw pictures).

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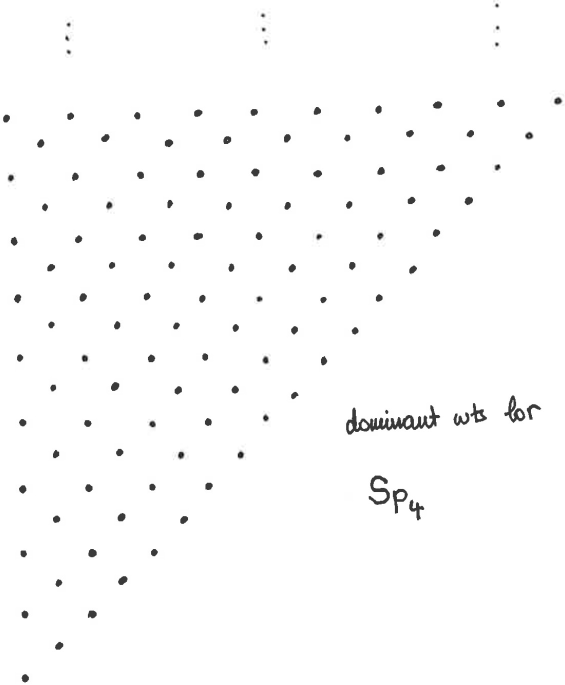
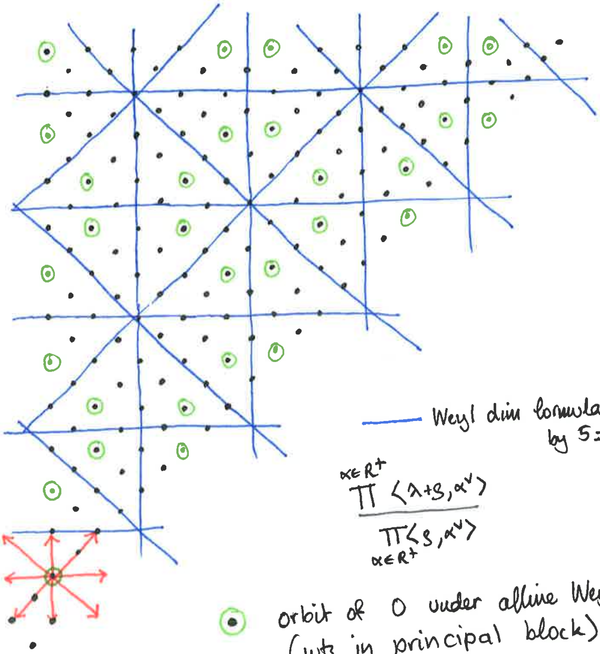
If k is of characteristic 0 then $\mathrm{Rep}\, G$ looks "just like representations of a compact Lie group". In positive characteristic one still has a classification of simple modules via highest weight, character theory etc. However the simple modules are usually much smaller than in characteristic zero.



dominant wts for

Sp_4

p=5



Weyl dim formula divisible by $5 = p$.

$$\frac{\prod_{\alpha\in R^+}\langle\lambda+\rho,\alpha^\vee\rangle}{\prod_{\alpha\in R^+}\langle\rho,\alpha^\vee\rangle}$$

orbit of 0 under affine Weyl group (wts in principal block).

$$\mathrm{Rep}_0 \overset{\oplus}{\subset} \mathrm{Rep}\, G \text{ the principal block.}$$

$$\mathrm{Rep}_0 \subset \mathrm{Rep}\, G \text{ depends on } p!$$

The analogue of the Kazhdan-Lusztig conjecture in this setting is:

Lusztig's character formula (1979): If $x \cdot 0$ is "restricted" (all digits in fundamental weights less than p) then

$$\mathrm{ch}(x \cdot_p 0) = \sum_y (-1)^{\ell(y)-\ell(x)} P_{w_0 y, w_0 x}(1) \mathrm{ch}(\Delta(y \cdot_p 0)).$$

For non-trivial reasons this gives a character formula for all simple modules.

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4. 2013: Building on work of Soergel and joint work with Elias, He, Kontorovich and Mcnamara I showed that the Lusztig conjecture *does not hold* for many p which grow exponentially in n. (E.g. fails for $p = 470\,858\,183$ for SL_{100}.)

$$\mathrm{Rep}_0 \overset{\oplus}{\subset} \mathrm{Rep}\, G \text{ the principal block.}$$

On Rep_0 one has the action of *wall-crossing functors*:

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Let W denote the affine Weyl group and $S = \{s_0, \ldots, s_n\}$ its simple reflections. For each $s \in S$ one has a wall-crossing functor Ξ_s. These generate the category of wall-crossing functors.

$$\langle \Xi_{s_0}, \Xi_{s_1}, \ldots, \Xi_{s_n} \rangle \circlearrowright \mathrm{Rep}_0 \, .$$

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The category of diagrammatic Soergel bimodules is a fundamental monoidal category in representation theory.

It can be thought of as one of the promised objects which has interesting 2-representation theory.

Hecke category for $W = \tilde{A}_2$

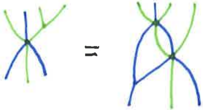
Generators: (objects) words in generators, thought of as coloured sequences:

Generators: (Morphisms) isotopy classes of (linear combinations of) finite planar graphs; generated by

(for all colours) (for all pairs of colours)

Relations:

(for all colours)



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Theorem: Our conjecture holds for $G = \mathrm{GL}_n$.

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Consequences of the conjecture...

The category of diagrammatic Soergel bimodules is a natural home for the canonical basis and Kazhdan-Lusztig polynomials. In fact, because it is defined over $\mathbb{Z}$ we get the *p-canonical basis* and *p-Kazhdan-Lusztig polynomials* for all p.

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Theorem: Assume our conjecture or $G = GL_n$. Then there exist simple formulas for the irreducible (if $p > 2h - 2$) and tilting (if $p > h$) characters in terms of the p-canonical basis.

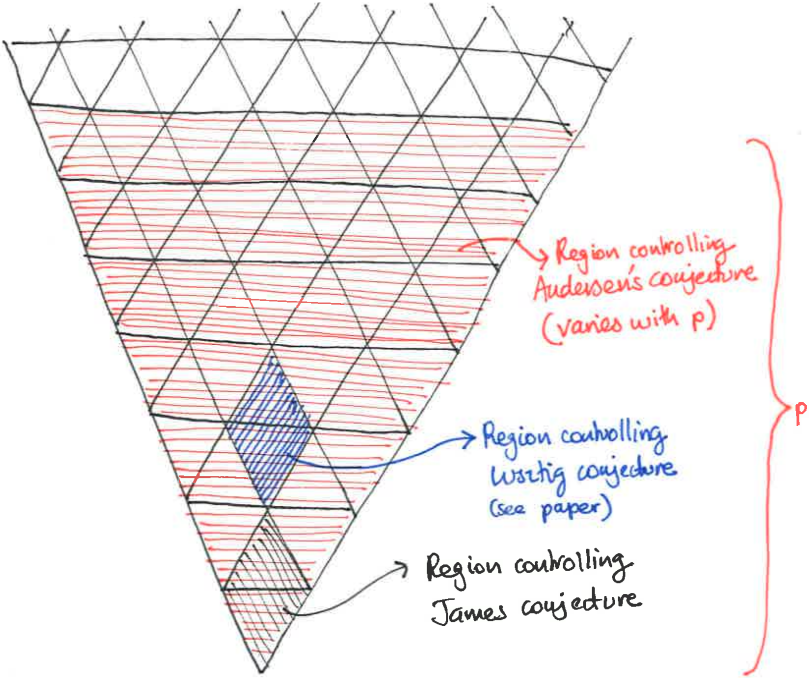
Thus the p-canonical basis controls precisely when Lusztig's conjecture holds, and tells us what happens when it fails.

Other consequences of our conjecture:

1. A complete description of Rep_0 in terms of the Hecke category, existence of a $\mathbb{Z}$-grading, etc.

Other consequences of our conjecture:

1. A complete description of Rep_0 in terms of the Hecke category, existence of a $\mathbb{Z}$-grading, etc.
2. All three main conjectures in this area (Lusztig conjecture, Andersen conjecture, James conjecture) are all controlled by the p-canonical basis. (Actually, the links to the James conjecture need some other conjectures. They should follow from work in progress by Elias-Losev.)


Region controlling Andersen's conjecture (varies with p)
Region controlling Lusztig conjecture (see paper)
Region controlling James conjecture
p

Thanks!

Slides:

people.mpim-bonn.mpg.de/geordie/Cordoba.pdf

Paper with Riche (all 136 pages!):

Tilting modules and the p-canonical basis,

http://arxiv.org/abs/1512.08296