Challenges in the representation theory of finite groups

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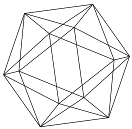


Hausdorff Kolloquium,
July 2016.

Let G be a group and V be a vector space.

A *representation* of G is a homomorphism:

$$\rho : G \to GL(V)$$

After fixing a basis of V we are "representing" our group by matrices.

Let G be a group and V be a vector space.

A *representation* of G is a homomorphism:

$$\rho : G \to GL(V)$$

A representation is the same thing as a G-*module*:

a linear action of G on V.

Representation theory is the study of linear actions (of groups, algebras, Lie algebras, . . .)

"I've spent most of the last five years thinking about what a representation is. I think I now understand, and I'm hoping both to write some of it down, and to begin thinking about what a group is. We can hope..."

– Ian Grojnowski, c. 2003.

For some of the results, see Grojnowski's entry on representation theory in the *The Princeton Companion to Mathematics*.

Mathematicians first began studying finite groups in earnest following the work of Galois in 1832.

Frobenius discovered the character table of a finite group in 1896 in Berlin. It took him another year to realise that he was studying representations.

Cayley's dictum that "a group is defined by means of the laws of combination of its symbols" would imply that, in dealing purely with the theory of groups, no more concrete mode of representation should be used than is absolutely necessary. It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

– Burnside, *Theory of groups of finite order*, 1897.
(One year after Frobenius' discovery of the character table.)

PREFACE TO THE SECOND EDITION

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is accordingly in the present edition a large amount of new matter.

– Burnside, *Theory of groups of finite order*, Second edition, 1911.
(15 years after Frobenius' discovery of the character table.)

1. A representation $\rho : G \to V$ is *simple* if the only G-invariant subspaces of V are $\{0\}$ and V itself.
2. A representation is *semi-simple* if it is isomorphic to a direct sum of simple representations.
3. If G is finite then any representation over $\mathbb{C}$ is semi-simple.
4. If $\rho : G \to V$ is a finite-dimensional representation over a field k then its character is the function

$$\begin{aligned} \chi_\rho : G &\to k \\ g &\mapsto \mathrm{Tr}(g). \end{aligned}$$

5. If k is a field of characteristic p then there exist representations over k which are not semi-simple if and only if p divides $|G|$. In the case the study of representations over k is called *modular representation theory*.

Recall the classification of finite simple groups:

$$\text{Cyclic} \quad \cup \quad \text{Alternating} \quad \cup \quad \text{Lie type} \quad \cup \quad \text{Sporadic}$$

$$\text{Cyclic} = \{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\}$$

$$\text{Alternating} = \{A_n \mid n \geqslant 5\}$$

$$\text{Lie type} = \text{simple groups of Lie type, e.g. } PSL_n(\mathbb{F}_q), \ldots, E_8(\mathbb{F}_q).$$

$$\text{Sporadic} = \text{the 26 sporadic simple } M_{11}, \ldots, \text{Monster}$$

The representation theory of the cyclic groups is easy.

We will concentrate on the representation theory of groups of Lie type and of symmetric groups.

Why the symmetric group S_n?

1. S_n might be the most "basic" of all finite groups;
2. S_n contains A_n as a normal subgroup of index 2;
3. S_n is the Weyl group of GL_n, and "is" $GL_n(\mathbb{F}_1)$!

(The sporadic simple groups will be ignored for the rest of this lecture.)

What is known?

Let us first consider the "easy" case of representations over $\mathbb{C}$:

Frobenius classified all simple representations of the symmetric group and computed their characters in 1900.

For groups of Lie type the situation is very intricate. However we know all simple representations and many of their character values. This has achieved by a number of authors, with the bulk of the work carried out by Lusztig (c. 1974-present).

If k is a field of characteristic p and $p \leqslant n$ (i.e. if p divides $|S_n| = n!$) then the study of representations of S_n over k ("modular representation theory of the symmetric group") is very complicated.

We know how many simple S_n-modules there are.

Except for a small number of cases, even their dimensions are *completely unknown!*

From now on: p is a prime, $q = p^r$ is a prime power and $\mathbb{F}_q$ is the finite field with q elements.

Consider $G(\mathbb{F}_q)$ a finite group of Lie type. For example we could take $G = GL_n(\mathbb{F}_q), Sp_{2n}(\mathbb{F}_q), SO_{2n+1}(\mathbb{F}_q), \dots$

We will consider the representation theory of $G(\mathbb{F}_q)$ over a field of characteristic p ("natural characteristic").

Example: $SL_2(\mathbb{F}_q)$ has a natural 2-dimensional representation: $\mathbb{F}_q^2$. This is a representation in natural characteristic.

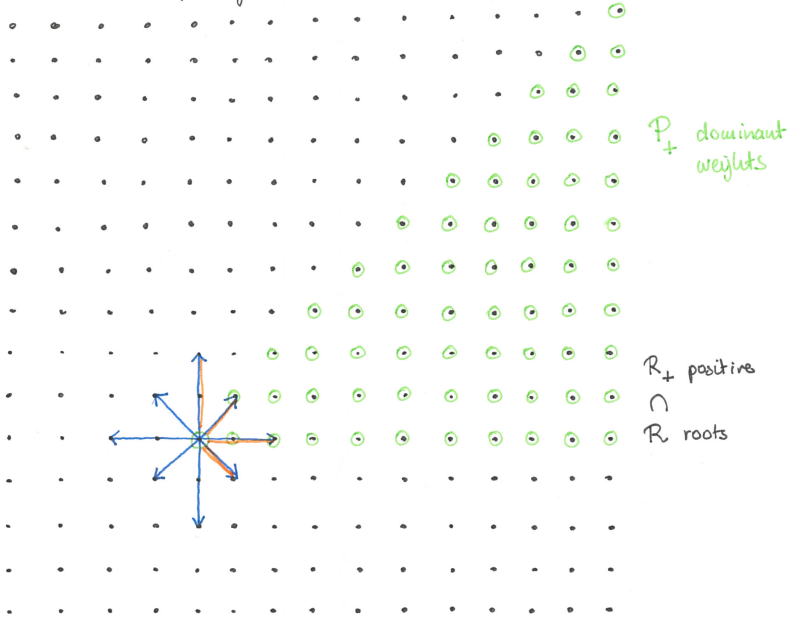
The smallest non-trivial representation of $SL_2(\mathbb{F}_q)$ over $\mathbb{C}$ is (usually) of much larger dimension (usually $\frac{q-1}{2}$).

There are surprising parallels between representations in natural characteristic the representation theory of compact Lie groups.

Let K denote a simply connected compact Lie group

(e.g. $K = SU_n$ or Sp_{2n}).

P weight lattice $\subset (\mathrm{Lie}\, T)^*$, $T \subset K$ maximal torus



Basic facts about the smooth representations of K:

1. Any smooth representation is semi-simple.
2. $$\begin{aligned} P_+ &\xrightarrow{\sim} \{\text{simple smooth reps of } K\}/\cong \\ \lambda &\mapsto V(\lambda) \end{aligned}$$
3. We have the Weyl character formula and Weyl dimension formula. For example:
$$\dim V(\lambda) = \prod_{\alpha \in R_+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}$$
where $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$.

A basic tool in this talk is the passage

$$K \leftrightsquigarrow G_{\mathbb{C}} \leftrightsquigarrow G_{\mathbb{Z}} \rightsquigarrow G_{\mathbb{F}_p}$$

$G_{\mathbb{Z}}$ is the "Chevalley scheme".

"semi-simple groups can be written down over $\mathbb{Z}$"

With some care this works also on representations:

$$V \leftrightsquigarrow V_{\mathbb{C}} \leftrightsquigarrow V_{\mathbb{Z}} \rightsquigarrow V_{\mathbb{F}_p}$$

Carrying out this process (again care is needed) allows one to associate to any $\lambda \in P_+$ an algebraic representation $\nabla(\lambda)$ of $G_{\mathbb{F}_p}$.

Here *algebraic* means that the matrix coefficients of

$$\rho : G_{\mathbb{F}_p} \to GL(V)$$

are regular functions on $G_{\mathbb{F}_p}$.

Example: Consider the natural representation of SL_2 on $V = kx \oplus ky$ (column vectors).

For any $m \geqslant 0$ we get a representation on the symmetric power

$$\nabla(m) := S^m(V)$$

(i.e. homogenous polynomials in x, y of degree m).

If $k = \mathbb{C}$ these restrict to yield all simple SU_2-modules.

These are not all simple in characteristic p:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x^p = (ax + cy)^p = a^p x^p + c^p y^p.$$

Hence $L(p) := kx^p \oplus ky^p \subset S^p(V)$ is a submodule.

The weird and wonderful world of rational representations:

In fact, $\nabla(p)/L(p)$ is simple and isomorphic to $L(p-2) := \nabla(p-2)$. Thus we have a short exact sequence

$$L(p) \hookrightarrow \nabla(p) \twoheadrightarrow L(p-2).$$

In the Grothendieck group we can write

$$[\nabla(p)] = [L(p)] + [L(p-2)]$$

Moreover, $L(p) \cong V^{(1)}$, where $V^{(1)}$ is the representation given by the Frobenius map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$$

Theorem (Chevalley): $\nabla(\lambda)$ contains a unique simple subrepresentation $L(\lambda)$. The $L(\lambda)$ are pairwise non-isomorphic and exhaust all simple G-modules.

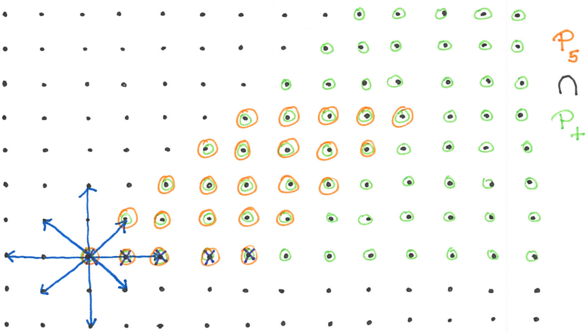
Hence one has a classification by highest weight just as in characteristic zero. However the simple modules are usually much smaller than in characteristic zero. (The definition of $L(\lambda)$ as a simple submodule is not explicit.)

Given any representation of $G_{\mathbb{F}_p}$ we can restrict to obtain a representation of the finite group $G(\mathbb{F}_q)$.

Theorem (Steinberg): There exists an explicit finite subset of "q-restricted weights" $P_q \subset P_+$ such that restriction gives a bijection

$$\{L(\lambda) \mid \lambda \in P_q\} \xrightarrow{\sim} \{\text{ simple } G(\mathbb{F}_q)\text{-modules }\}/\cong$$

Thus understanding algebraic representations of $G_{\mathbb{F}_p}$ also answers the question for the finite group $G(\mathbb{F}_q)$ in natural characteristic.



Explicit constructions of $L(\lambda)$ are a distant dream (except for SL_2).

Instead we try to write the unknown in terms of the "known":

$$[L(\lambda)] = \sum a_{\mu\lambda}[\nabla(\mu)].$$

As "reductions modulo p", the $[\nabla(\mu)]$ have the same dimensions and characters as their characteristic zero cousins (Weyl's character formula). One can see the above equality as an identity of formal characters.

Example: For SL_2 we saw

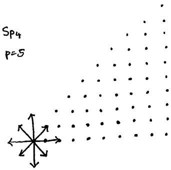
$$[L(p)] = [\nabla(p)] - [\nabla(p-2)]$$

$SL_2, \; p=5$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Dimension
L(0)	1															1
L(1)		1														2
L(2)			1													3
L(3)				1												4
L(4)					1											5
L(5)				-1		1										2
L(6)			-1				1									4
L(7)		-1						1								6
L(8)	-1								1							8
L(9)										1						10
L(10)	2								-1		1					3
L(11)		2						-1				1				6
L(12)			2				-1						1			9
														1		12

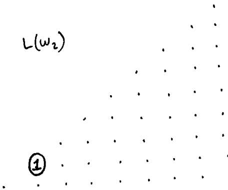
Sp_4

$p=5$

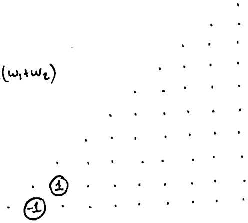


$L(0)$

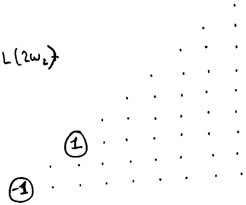
$L(\omega_2)$



$L(\omega_1+\omega_2)$

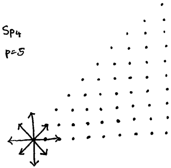


$L(2\omega_2)$

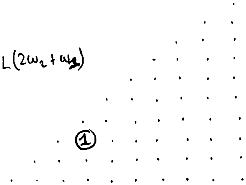


Sp_4

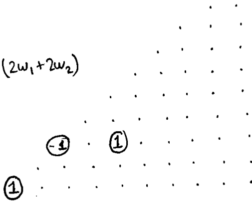
$p=5$



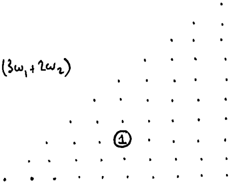
$L(2\omega_2+\omega_1)$



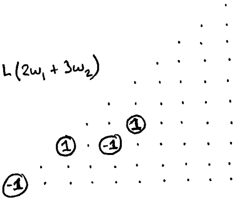
$L(2\omega_1+2\omega_2)$



$L(3\omega_1+2\omega_2)$

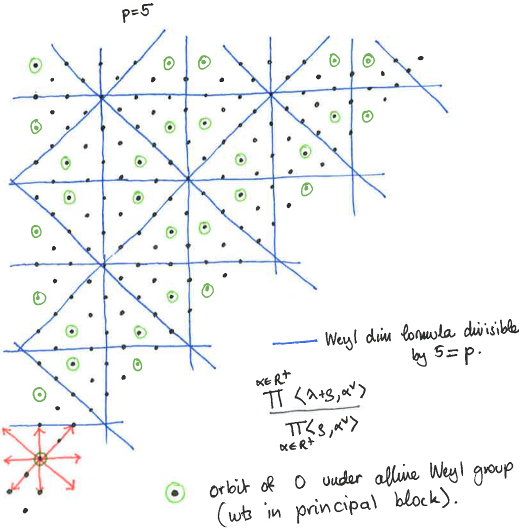


$L(2\omega_1+3\omega_2)$



Verma noticed that behind all of this lurks an action of an affine Weyl group "dilated by p".

We denote the action of this group $\lambda \mapsto x \cdot_p \lambda$.



SL_2, $p=5$

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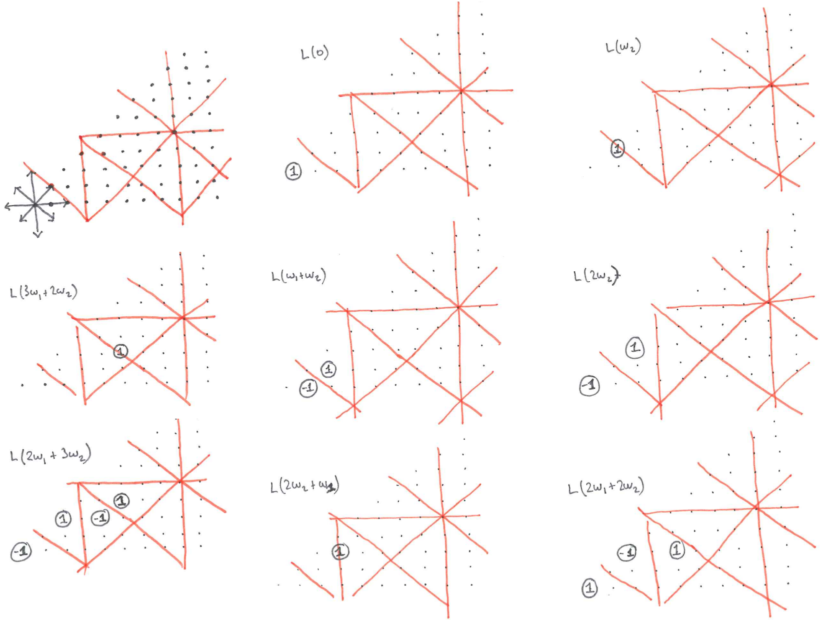
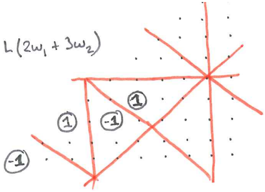
L(8)	-1								1							8
L(9)										1						10
L(10)	1								-1		1					3
L(11)		1						-1				1				6
L(12)			1				-1						1			9
L(13)				1		-1								1		12
L(14)															1	15


L(0)
L(ω₂)
L(3ω₁+2ω₂)
L(ω₁+ω₂)
L(2ω₂)
L(2ω₁+3ω₂)
L(2ω₂+ω₁)
L(2ω₁+2ω₂)



Lusztig's character formula (1980): If $x \cdot_p 0$ is "p-restricted" and p is "not too small" then

$$[L(x \cdot_p 0)] = \sum_y (-1)^{\ell(y)-\ell(x)} P_{w_0 y, w_0 x}(1) [\nabla(y \cdot_p 0)].$$

The $P_{x,y}$ are Kazhdan-Lusztig polynomials associated to the affine Weyl group. (Tricky, combinatorial, but "easy" if you or your computer has a good memory.)

This formula is enough to determine all characters.

Lusztig's conjecture (1980).

Proceedings of Symposia in Pure Mathematics
Volume 37, 1980

SOME PROBLEMS IN THE REPRESENTATION THEORY OF FINITE CHEVALLEY GROUPS

GEORGE LUSZTIG[1]

obtained by reducing modulo p the irreducible representation with highest weight $-w\rho - \rho$ of the corresponding complex group. (It is well defined in the Grothendieck group.) We assume that $\alpha_0^\vee(\rho) < p$.

Problem IV. *Assume that w is dominant and it satisfies the Jantzen condition $\alpha_0^\vee(-w\rho) \leqslant p(p-h+2)$, where h is the Coxeter number. Then*

$$\operatorname{ch} L_w = \sum_{\substack{y \in W_a,\ \text{dominant} \\ y \leqslant w}} (-1)^{l(w)-l(y)} P_{y,w}(1) \operatorname{ch} V_y. \tag{4}$$

From this, one can deduce the character formula for any irreducible finite dimensional representation of G (over $\overline{\mathbf{F}}_p$), by making use of results of Jantzen and Steinberg. The evidence for this character formula is very strong. I have verified it in the cases where G is of type A_2, B_2 or G_2. (In these cases, ch L_w has been computed by Jantzen.) One can show using results of Jantzen [**2**, Anhang]

Understanding Lusztig's conjecture, and in particular deciding for which p it holds has been one of the central puzzles in modular representation theory over the last thirty years.

What "large" means on the previous slide is a tricky business.

Let h denote the Coxeter number of G

(e.g. $h = n$ for GL_n, $h = 2n$ for SP_{2n}, $h = 30$ for E_8):

1. 1980: Lusztig conjectured $p \geqslant 2h - 3$ (Jantzen condition);
2. 1985: Kato conjectured $p \geqslant h$;
3. 1994: Several hundred pages of Andersen-Jantzen-Soergel, Kazhdan-Lusztig, Kashiwara-Tanisaki and Lusztig prove the conjecture for large p without any explicit bound!

W. Soergel (2000): "Bei Wurzelsystemen verschieden von A_2, B_2, G_2, A_3, weiß man aber für keine einzige Charakteristik ob sie hinreichend groß ist."
…a particularly strange situation for finite group theorists.

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3. 1994: Andersen-Jantzen-Soergel, Kazhdan-Lusztig, Kashiwara-Tanisaki, Lusztig: the conjecture holds for large p;
4. 2008: Fiebig gave an explicit enormous bound (e.g. $p > 10^{40}$ for SL_9 against the hoped for $p \geqslant 11$)!

The following 2013 theorem has a part joint with Xuhua He and another part joint with Alex Kontorovich and Peter McNamara, and builds on work done in a long term project with Ben Elias.

Theorem

There exists a constants $a > 0$ and $c > 1$ such that Lusztig's conjecture on representations of SL_n fails for many primes $p > ac^n$ and $n \gg 0$.

The theorem implies that there is no polynomial bound in the Coxeter number for the validity of Lusztig's conjecture. This should be compared with the hope (believed by many for over thirty years) that the bound is a simple linear function of Coxeter number.

Provably we can take $a = 5/7$ and $c = 1.101$. Experimentally c can be taken much larger. For example, Lusztig's conjecture fails for $SL_{100}(\mathbb{F}_p)$ with $p = 470\ 858\ 183$.

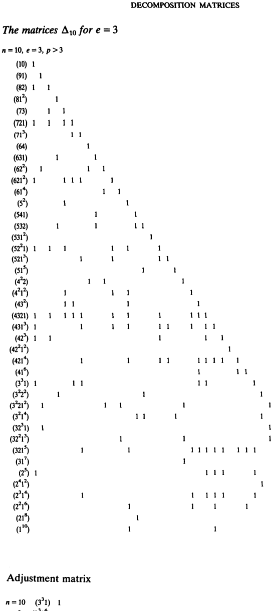
These results also yield counter-examples to the James conjecture (1990).

Gordon James formulated his conjecture following formidable calculations. He conjectured a formula for the decomposition numbers of simple representations of S_n if $p > \sqrt{n}$ ("p not too small").

His conjecture, if true, would represent major progress on the problem.

His conjecture is true for $n = 1, 2, \ldots, 22$.

James, *The decomposition matrices of $GL_n(q)$ for $n \leqslant 10$*, Proc. London Math. Soc. (3) 60 (1990), no. 2, 225–265.



Following a line of attack suggested by Joe Chuang, the previous result also yields:

Theorem

The James conjecture fails "generically". In particular, it is not true for S_n for all $n \geqslant 1\,744\,860$.

We are trying to work out where, between $n = 22$ and $n = 1\,744\,860$, the conjecture first goes wrong.

There is still much to say about S_n, possibly the most fundamental of all finite groups ...

A key step in establishing this theorem is a "translation of the problem into topology" completed by Wolfgang Soergel in 2000.

This is an instance of "geometric representation theory": the topology of complex algebraic varieties has much to say about representation theory.

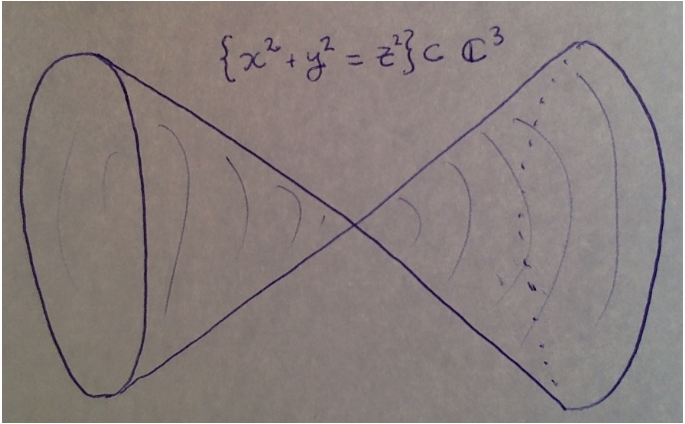
This field has been driven by Lusztig and many others over the past forty years. It must sadly stay a black box in this talk.

Example: The characters of $GL_n(\mathbb{F}_q)$ may be described via certain geometric objects ("character sheaves") which live on of $GL_n(\mathbb{C})$. Thus there is a geometric procedure to produce the character table of $GL_n(\mathbb{F}_q)$ for "all q's at once".

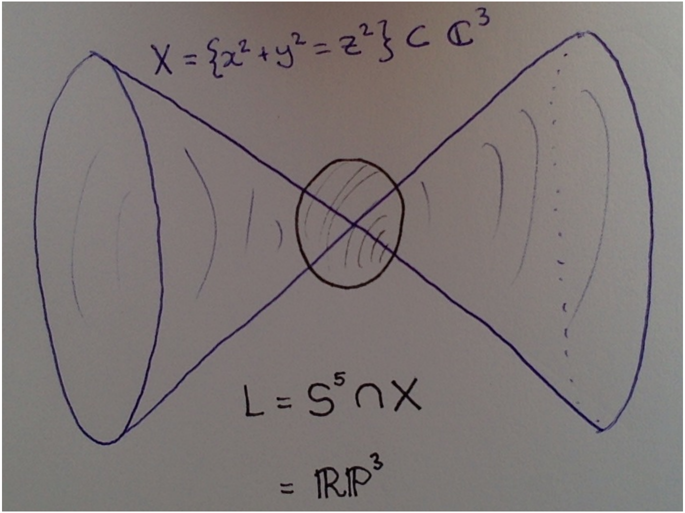
Roughly speaking, the coefficients where one takes representations corresponds to the coefficients of cohomology.

Example:

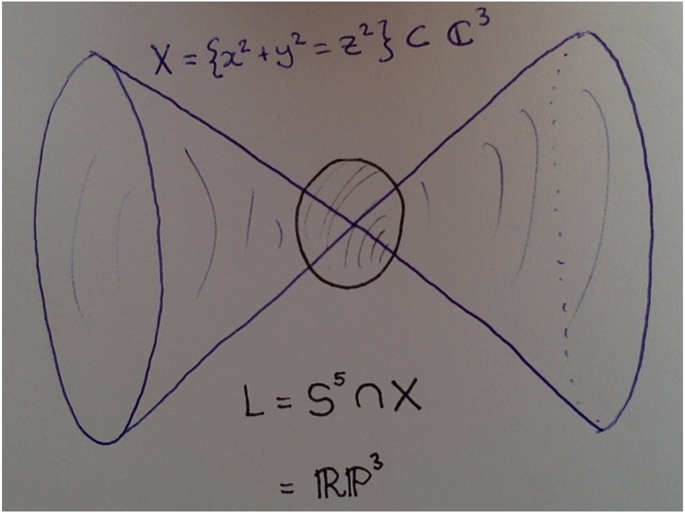
Consider the quadric cone ($\dim_{\mathbb{C}} = 2$, singular space). We can draw a real picture:



If we intersect a small sphere around the singularity with X we obtain ...



Hint: $X = \mathbb{C}^2/(\pm 1)$ so $L = S^3/(\pm 1) = \mathbb{RP}^3$.



We have $H^2(\mathbb{RP}^3) = \mathbb{Z}/2\mathbb{Z}$ and all other groups are torsion free. This turns out to be *equivalent* to the fact that the representation theory of S_2 is “different” in characteristic 2.

Theorem

Let c be a non-zero entry of a word w of length ℓ in the generators:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

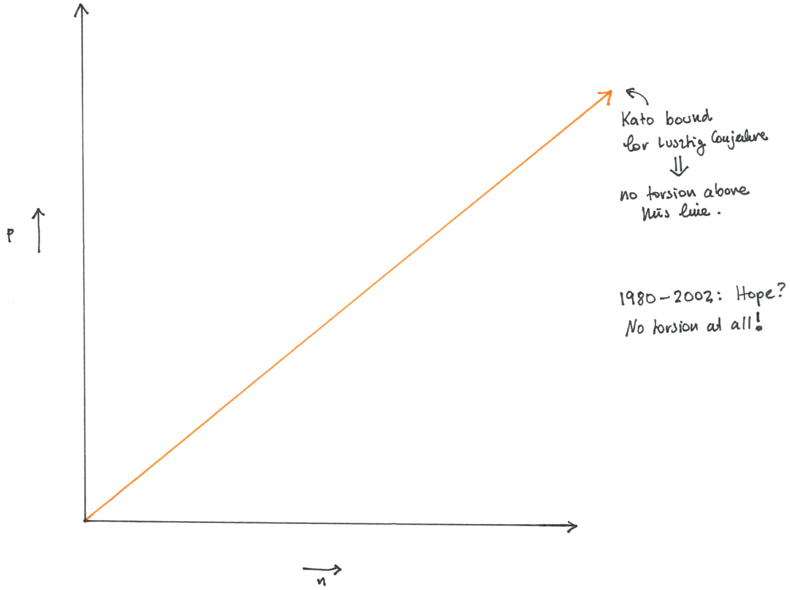
Then associated to w one can find $\mathbb{Z}/c\mathbb{Z}$-torsion in a variety controlling the representation theory of $SL_{3\ell+5}$.

In particular, any prime p dividing c which is larger than $3\ell + 5$ gives a counter-example to the expected bounds in Lusztig's conjecture.

For the experts: we find torsion in the costalk of an integral intersection cohomology complex of a Schubert variety in $SL_{3\ell+5}(\mathbb{C})/B$.

Non-trivial number theory (relying on ideas surrounding the affine sieve and Zaremba's conjecture) yields that the prime divisors of c above grow like $O(c^n)$ for come $c > 1$.

p-torsion in local intersection cohomology of Schubert varieties in SL_n/B.



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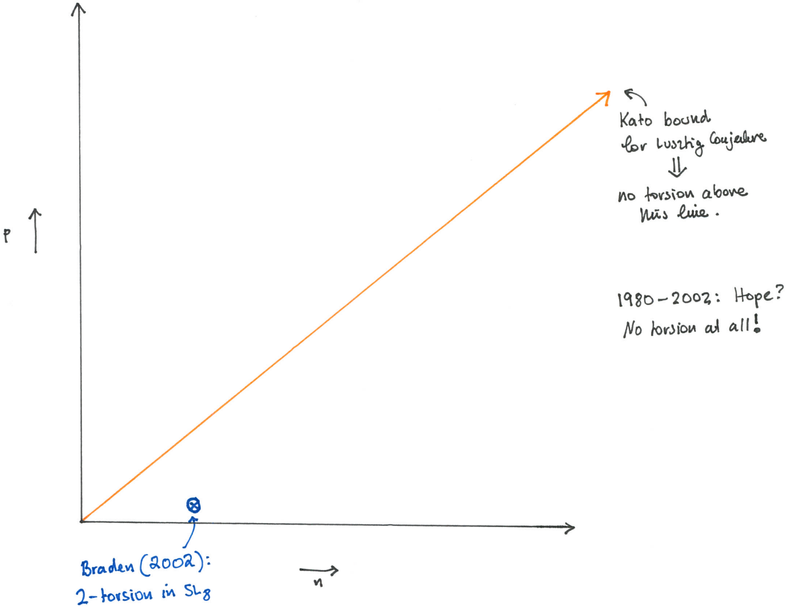

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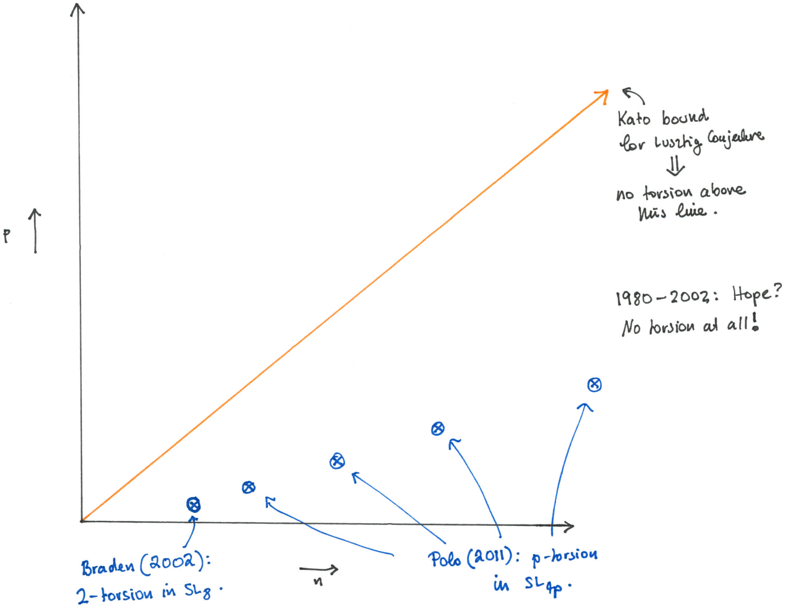

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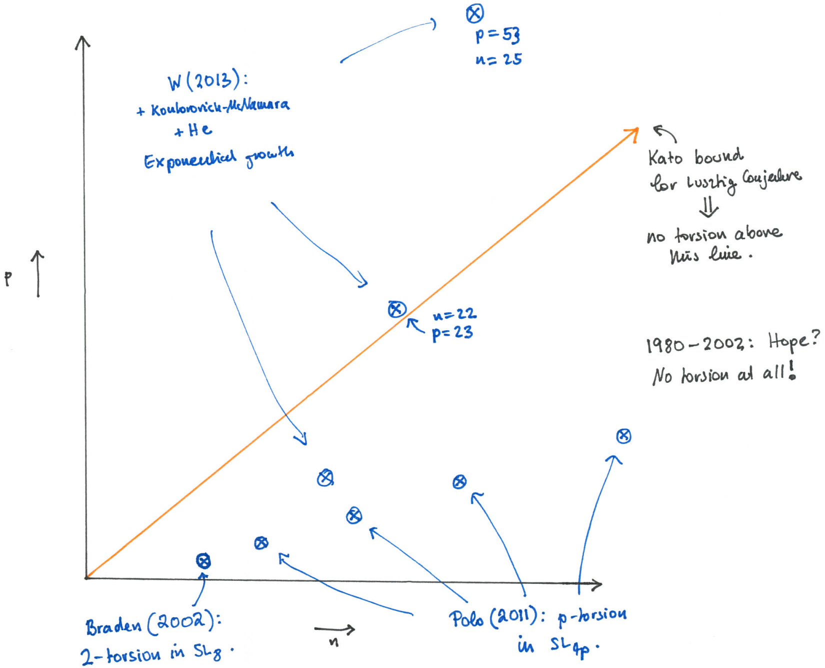

In summary:

The Lusztig and James conjecture predict a remarkable regularity in the modular representation theory of symmetric groups and finite groups of Lie type for large primes.

However it takes much longer for this regularity to show itself than was expected.

For "mid range primes" (e.g. $n < p < c^n$) subtle and unexpected arithmetic questions show up in the representation theory of groups like $GL_n(\mathbb{F}_p)$.

In recent joint work with Simon Riche we have proposed a new conjecture which gives an answer for all primes. Very roughly, it involves replacing the Kazhdan-Lusztig polynomials in Lusztig's conjecture with p-Kazhdan-Lusztig polynomials. Unfortunately these polynomials are much more difficult to compute.

Our conjecture is true for GL_n and SL_n if $p > 2n - 2$ and in work in progress with Achar, Riche and Makisumi we hope to prove it for all G.

However we still can't decide exactly where the uniformity of the Lusztig and James conjecture takes over.

However in spite of all our efforts, we know very little about finite groups. The mystery has not been resolved, we cannot even say for sure whether order or chaos reigns. If any excitement can be derived from what I have to say, it should come from the feeling of being at a frontier across which we can see many landmarks, but which as a whole is unexplored, of planning ways to find out about the unknown, even if the pieces we can put together are few and far apart. My hope then is that some of you may go out with the idea: "Now let me think of something better myself."

– Richard Brauer, *On finite groups and their characters*,

Bull. Amer. Math. Soc. Volume 69, Number 2 (1963), 125-130.