Mathematics in light of representation theory

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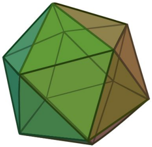


BMS Friday Colloquium.

Symmetry and the notion of a group: 1832 - 1920.

Symmetry is all around us.

We have been observing and studying symmetry for millennia.

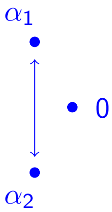
The set of symmetries form a *group*.

We owe the term *group(e)* to Galois (1832).

Galois theory:

$f \in \mathbb{Q}[x]$ $\quad$ $x^2 + x + 1 = \frac{x^3-1}{x-1}$

$\{\alpha_i\}$ roots of f



Form $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ $\quad$ $\mathbb{Q}(e^{2\pi i/3})$.

$\mathrm{Gal}(K, \mathbb{Q}) := \mathrm{Aut}(\mathbb{Q}(\alpha_1, \ldots, \alpha_m))$ ("Galois group")

$\mathrm{Gal}(K, \mathbb{Q})$ acts on $\{\alpha_1, \ldots, \alpha_m\}$.

Galois theory: This action tells us everything about f and its roots.

En d'autres termes, quand un groupe G en contient un autre H, le groupe G peut se partager en groupes, que l'on obtient chacun en opérant sur les permutations de H une même substitution ; en sorte que

$$G = H + HS + HS' + \dots.$$

1. Écrite la veille de la mort de l'auteur. (Insérée en 1832 dans la *Revue encyclopédique*, numéro de septembre, page 568.) (J. LIOUVILLE.)

Et aussi il peut se diviser en groupes qui ont tous les mêmes substitutions, en sorte que

$$G = H + TH + T'H + \dots.$$

Ces deux genres de décompositions ne coïncident pas ordinairement. Quand ils coïncident, la décomposition est dite *propre*.

Il est aisé de voir que, quand le groupe d'une équation n'est susceptible d'aucune décomposition propre, on aura beau transformer cette équation, les groupes des équations transformées auront toujours le même nombre de permutations.

Au contraire, quand le groupe d'une équation est susceptible d'une décomposition propre, en sorte qu'il se partage en M groupes de N permutations, on pourra résoudre l'équation donnée au moyen de deux équations : l'une aura un groupe de M permutations, l'autre un de N permutations.

Lors donc qu'on aura épuisé sur le groupe d'une équation tout ce qu'il y a de décompositions propres possibles sur ce groupe, on arrivera à des groupes qu'on pourra transformer, mais dont les permutations seront toujours en même nombre.

Si ces groupes ont chacun un nombre premier de permutations, l'équation sera soluble par radicaux ; sinon, non.

$H \subset G$ is a subgroup

Letter to Auguste Chevalier in 1832

written on the eve of Galois' death

notion of a normal subgroup

notion of a simple group

notion of a soluble group

main theorem of Galois theory

On the Theory of Groups. 41

be noticed also, that if $\theta=\phi$, then, whatever the symbols α, β may be, $\alpha\theta\beta=\alpha\phi\beta$, and conversely.

A set of symbols,

$$1,\ \alpha,\ \beta\ldots$$

all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself belongs to the set, is said to be a *group**. It follows that if the entire group is multiplied by any one of the symbols, either as further or nearer factor, the effect is simply to reproduce the group; or what is the same thing, that if the symbols of the group are multiplied together so as to form a table, thus:—

Further factors.

Nearer factors.	1	α	β	..
1	1	α	β	..
α	α	α^2	$\beta\alpha$	
β	β	$\alpha\beta$	β^2	
	:			

that as well each line as each column of the square will contain all the symbols $1,\ \alpha,\ \beta$ . . It also follows that the product of any number of the symbols, with or without repetitions, and in any order whatever, is a symbol of the group. Suppose that the

Cayley, *On the theory of groups, as depending on the symbolic equation* $\Theta^n = 1$,

Philosophical Magazine, 4th series, 7, 1854.

Cayley (1878): "A group is defined by the law of composition of its members."

I. Je zwei Elemente A und B bestimmen in der angegebenen Reihenfolge ***eindeutig*** ein drittes, welches mit AB bezeichnet wird.

$(g_1, g_2) \mapsto g_1 g_2,$

II. Aus jeder der beiden Gleichungen $AC = BC$ oder $CA = CB$ folgt $A = B$.

$gh = gh' \Rightarrow h = h'$
$hg = h'g \Rightarrow h = h'$

III. Für die Operation, durch welche AB aus A und B entspringt, gilt das ***associative*** Gesetz $(AB)C = A(BC)$, aber nicht nothwendig das ***commutative*** Gesetz $AB = BA$.

$g_1(g_2 g_3) = (g_1 g_2) g_3,$

IV. Die Anzahl der Elemente ist endlich.

G is finite.
$\Rightarrow$ existence of inverses

– Frobenius, *Neuer Beweis des Sylowschen Satzes*, Crelle, 1884.

CHAPTER II.

THE DEFINITION OF A GROUP.

12. In the present chapter we shall enter on our main subject and we shall begin with definitions, explanations and examples of what is meant by a group.

Definition. Let

$$A, B, C, \ldots$$

represent a set of operations, which can be performed on the same object or set of objects. Suppose this set of operations has the following characteristics.

(α) The operations of the set are all distinct, so that no two of them produce the same change in every possible application.

(β) The result of performing successively any number of operations of the set, say A, $B, \ldots$, K, is another definite operation of the set, which depends only on the component operations and the sequence in which they are carried out, and not on the way in which they may be regarded as associated. Thus A followed by B and B followed by C are operations of the set, say D and E; and D followed by C is the same operation as A followed by E.

(γ) A being any operation of the set, there is always another operation A_{-1} belonging to the set, such that A followed by A_{-1} produces no change in any object.

Burnside, *Theory of groups of finite order*, 1897.

Surely the definition of group is one of the most intuitive and useful in all of mathematics.

Why did it take us so long to realize the importance of this notion?

Speculation: In Galois theory we first see the importance of the *structure of the symmetry*, or what we now call group theory.

That is, we move from the usefulness of *one* symmetry to the study of the *set of all* symmetries.

It was Galois who first asked:

Is a finite group G simple?

A group G is *simple* if it has no non-trivial normal subgroups.

A subgroup $H \subset G$ is normal if it is the kernel of some homomorphism $G \to G'$.

Jordan-Hölder theorem: finite simple groups are the building blocks of all finite groups.

We also see this is Klein's Erlangen program (1872):

Geometry is its group of symmetries.

This idea also pervades 20^{th} and 21^{st} century theoretical physics.

Representation theory is the study of linear group actions:

A *representation* of a group G is a homomorphism

$$\rho : G \to GL(V)$$

for some vector space V.

A representation is the same thing as a linear action of G on V.

A representation is *irreducible* if the only subspaces $U \subset V$ which are stable under the action of G are $\{0\} \subset V$ and V itself.

There is a Jordan-Hölder theorem: the irreducible representations are the building blocks of all representations.

A representation theorist's strategy:

problem involving a group action $G \circlearrowright X$ ⇝ problem involving a linear group action $G \circlearrowright k[X]$

⇝ "decomposition" of problem $G \circlearrowright \bigoplus V_i$ ⇝ problem involving a group action $G \circlearrowright X$

Three examples of mathematics in light of representation theory

Example 1: Finite group actions on sets.

For a fixed finite group G these two problems are "the same":
1) classify finite sets with G-action;
2) classify subgroups $H \subset G$ up to conjugacy.

The equivalent problems turn out to be extremely complicated. Because every finite group is a subgroup of a symmetric group, a solution to (2) would be something like a classification of all finite groups. There are more than 30 papers on the classification of maximal subgroups of the monster simple group.

However the analogous linear problem "classify $\mathbb{C}$-vector spaces with linear G-action" is representation theory. Here we have a satisfactory answer for many groups.

Example 2: The circle and the Fourier transform.

Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Then S^1 is a (Lie) group.

For any $m \in \mathbb{Z}$ we have a one-dimensional representation of S^1 via:

$$S^1 \in z \mapsto z^m \in \mathbb{C}^* = GL_1(\mathbb{C}).$$

In fact, these are all irreducible representations of S^1!

Now we consider: $S^1 \circlearrowright S^1$.

We linearize this action and consider for example

$$S^1 \circlearrowright L^2(S^1, \mathbb{C}).$$

Now our irreducible characters z^m belong to the right hand side.

Moreover, as Hilbert spaces:

$$L^2(S^1, \mathbb{C}) = \hat{\bigoplus} \mathbb{C} z^m$$

If we identify $S^1 = \mathbb{R}/\mathbb{Z}$ then the functions z^m become the fundamental frequencies $\lambda \mapsto e^{2\pi i m \lambda}$ of Fourier analysis.

Moral: The decomposition of $L^2(S^1, \mathbb{C})$ into irreducible representations is the theory of Fourier series.

Similarly, the Fourier transform can be explained in terms of representations of $(\mathbb{R}, +)$, spherical harmonics in terms of representations of $SO(3) \circlearrowright S^2$, ...

Example 3: Rational points and Fermat's last theorem.

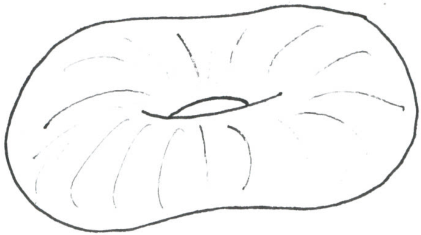
Suppose we want to find rational solutions to an equation X like:

$$y^2 = x^3 - x^2 - 24649x + 1355209$$

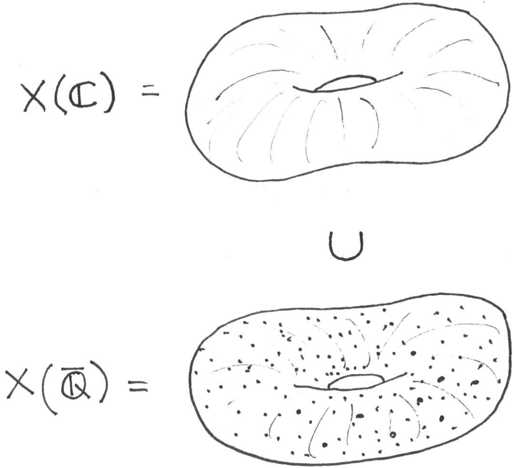
Let us write $X(\mathbb{C})$ for the solutions with $x, y \in \mathbb{C}$, $X(\mathbb{Q})$ for solutions $x, y \in \mathbb{Q}$ etc.

It turns out that $X(\mathbb{C})$ is a Riemann surface of genus one:

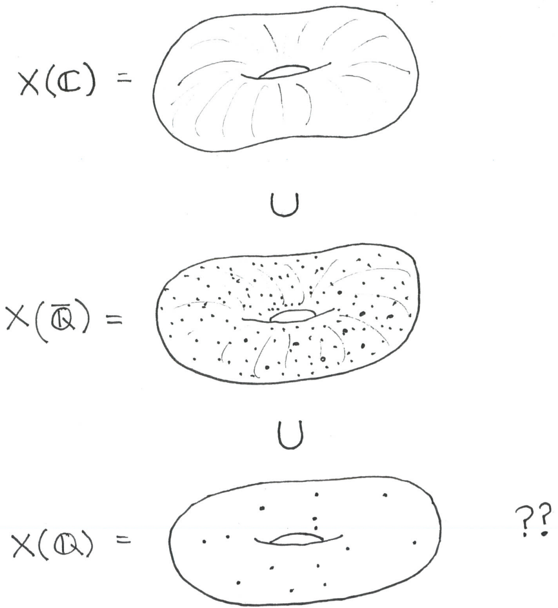
$X(\mathbb{C}) =$



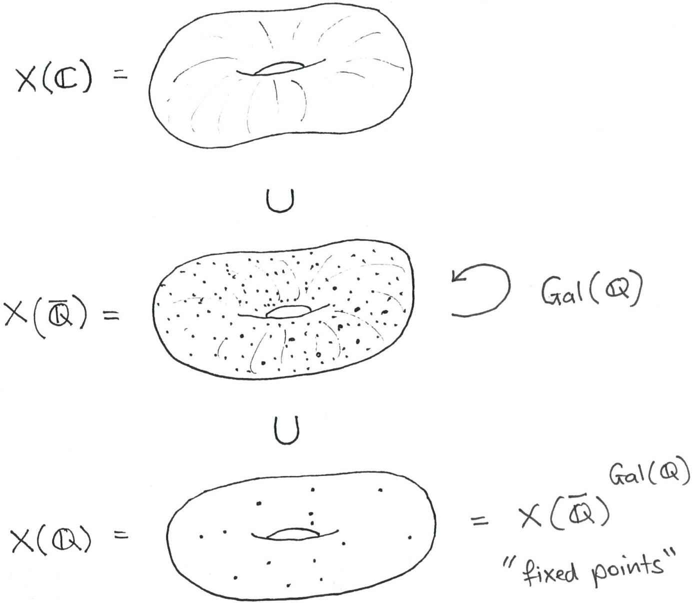
The points in an algebraic closure $X(\overline{\mathbb{Q}})$ are also "easy" (think of the stars in the night sky):



The tricky point is to find the rational points $X(\mathbb{Q})$:



Let $\mathrm{Gal}(\mathbb{Q})$ denote the absolute Galois group (automorphisms of $\mathbb{Q} \subset \overline{\mathbb{Q}}$). Group theory interpretation:



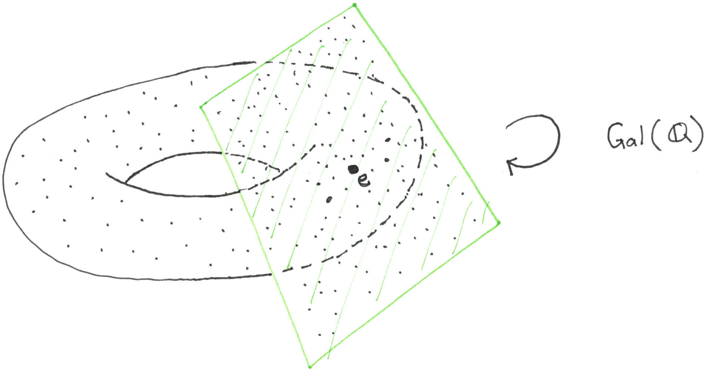
Diophantine geometry can be encoded in questions like:

Understand the $\mathrm{Gal}(\mathbb{Q})$-action on $X(\overline{\mathbb{Q}})$.

But we will probably never understand the $\mathrm{Gal}(\mathbb{Q})$ sets $X(\overline{\mathbb{Q}})$.

However representation theory suggests that we should cook up a linear object out of the action of $\mathrm{Gal}(\mathbb{Q})$ out of $X(\overline{\mathbb{Q}})$.

It turns out that we can do this, and it is *extremely* profitable. The short version: $\mathrm{Gal}(\mathbb{Q})$ acts in a very interesting way on $H_1(X; \mathbb{Q}_\ell) = \mathbb{Q}_\ell^2$. (Can be thought of as something like a tangent space.)



This is the structure behind the proof of Fermat's last theorem:

1. start with a solution $x^n + y^n = z^n$ with $x, y, z \in \mathbb{Z}$, $n > 2$;
2. build from this solution a strange elliptic curve E (the "Frey curve");
3. observe that such a curve would give a very strange G-representation $H_1(E; \mathbb{Q}_3)$ (Frey, Serre, Ribet);
4. show that such a G-representation cannot exist (Wiles, Taylor-Wiles).

Moreover the Langlands program gives us a vast array of theorems and conjectures linking representations of Galois groups coming from Diophantine problems (like the rational points question above) to analysis and automorphic forms.

A beautiful introduction to these ideas:

R. P. Langlands, *Representation theory: its rise and its role in number theory.* Proceedings of the Gibbs Symposium (New Haven, CT, 1989)

Representations of finite groups and the character table

Basic theorems in the representation theory of a finite group G:

1. any $\mathbb{C}$-representation of G is isomorphic to a direct sum of irreducible representations;
2. $$\#\left\{ \begin{array}{c} \text{irreducible} \\ \mathbb{C}\text{-representations of } G \end{array} \right\}_{/\cong} = \#\left\{ \begin{array}{c} \text{conjugacy} \\ \text{classes in } G \end{array} \right\}.$$
3. Any finite dimensional representation $\rho : G \to GL(V)$ is determined (up to isomorphism) by its *character*:
$$\chi_\rho : G \to \mathbb{C} : g \mapsto \operatorname{Tr} \rho(g).$$

Hence, we know (almost) everything about the $\mathbb{C}$-representations of a group once we know the characters of the irreducible representations of our group G.

$$\chi(hgh^{-1}) = \mathrm{Tr}(\rho(hgh^{-1})) = \mathrm{Tr}(\rho(h)\rho(g)\rho(h)^{-1}) = \mathrm{Tr}(\rho(g)) = \chi(g).$$

Hence χ is a function on the conjugacy classes of G.

All of this information can be conveniently displayed in the *character table* of G. The rows give the irreducible characters of G and the columns are indexed by the conjugacy classes of G.

The character table of G is the $\mathbb{C}$-linear shadow of G.

The first character table ever published. Here G is the alternating group on 4 letters, or equivalently the symmetries of the tetrahedron.

dnung 3 zwei inverse Classen (2) und $(3) = (2')$. Sei ρ eine prim

bische Wurzel der Einheit.

Tetraeder. $h = 12$.

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	h_α
χ_0	1	3	1	1	1
χ_1	1	-1	1	1	3
χ_2	1	0	ρ	ρ^2	4
χ_3	1	0	ρ^2	ρ	4

Die Werthe von χ_0 sind zugleich die von $f = e$.

Frobenius, *Über Gruppencharaktere*, S'ber. Akad. Wiss. Berlin, 1896.

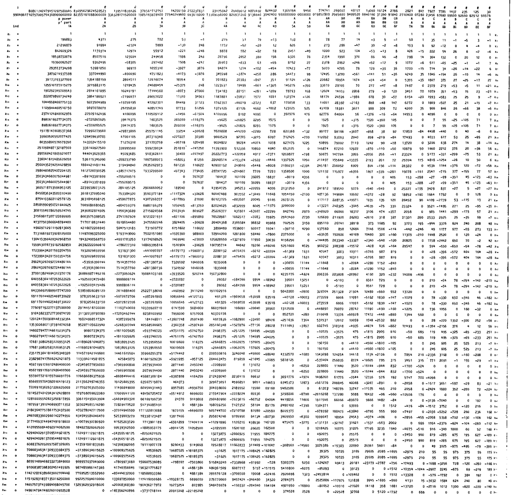
Now $G = S_5$, the symmetric group on 5 letters of order 120:

$h = 120$

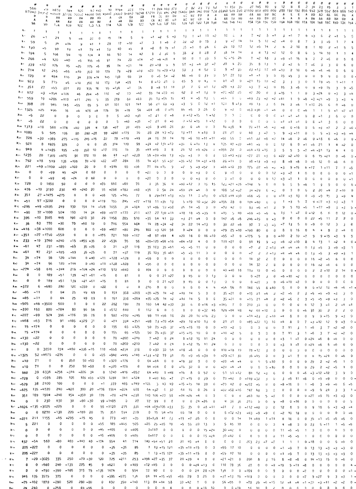
	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	$\chi^{(4)}$	$\chi^{(5)}$	$\chi^{(6)}$	h_α
χ_0	1	5	5	4	4	6	1	1
χ_1	1	1	1	0	0	−2	1	15
χ_2	1	1	−1	2	−2	0	−1	10
χ_3	1	−1	−1	1	1	0	1	20
χ_4	1	−1	1	0	0	0	−1	30
χ_5	1	0	0	−1	−1	1	1	24
χ_6	1	1	−1	−1	1	0	−1	20

Conway, Curtis, Norton, Parker, Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With computational assistance from J. G. Thackray.* Oxford University Press, 1985.

$M = F_1$



ind	1A	2A	2B	3A	3B	3C	4A	4B	4C	4D
p power		A	A	A	A	A	B	A	B	B
p' part		A	A	A	A	A	A	A	A	A
χ_1 +	1	1	1	1	1	1	1	1	1	1
χ_2 +	196883	4371	275	782	53	−1	275	51	19	−13
χ_3 +	21296876	91884	−2324	7889	−130	248	1772	−52	−20	12
χ_4 +	842609326	1139374	12974	55912	−221	−248	8878	782	−82	78
χ_5 +	18538750076	8507516	123004	249458	1598	248	28796	2652	180	156
χ_6 +	19360062527	9362495	−58305	297482	1508	−247	35903	−833	63	−65
χ_7 +	293553734298	53981850	98970	1055310	−3927	3876	94874	1274	−102	−454
χ_8 +	3879214937598	337044990	−690690	4751823	−4173	−3876	345598	−3874	−258	286
χ_9 +	36173193327999	1354188159	2864511	12616074	18954	0	701823	20383	−897	351
χ_{10} +	125510727015275	3215883115	1219435	24688459	−25375	248	1223531	19499	−661	−1365
χ_{11} +	190292345709543	2814161895	16249191	17144568	−8073	27000	734183	22151	−281	−1209
χ_{12} +	222879856734249	3864186921	−7196631	26057022	11310	34503	1187881	−39831	1577	1001
χ_{13} +	1044868466775133	9223504989	−15756195	47292301	84448	31123	1762397	−46019	−2723	637
χ_{14} +	1109944460516150	9697078070	26155830	40851749	97733	61256	1379126	87126	4662	−1002
χ_{15} +	2374124840062976	22509162496	4100096	110509112	−1456	−61504	4100096	8192	4096	0



But if you're not yet convinced you are not alone!

Cayley's dictum that "a group is defined by means of the laws of combination of its symbols" would imply that, in dealing purely with the theory of groups, no more concrete mode of representation should be used than is absolutely necessary. It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

– Burnside, *Theory of groups of finite order*, 1897.
(One year after Frobenius' definition of the character.)

PREFACE TO THE SECOND EDITION

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is accordingly in the present edition a large amount of new matter.

– Burnside, *Theory of groups of finite order*, Second edition, 1911.
(15 years after Frobenius' definition of the character table.)

What led Frobenius to his marvellous definition?

As well as an inherent fascination in finite groups, Frobenius was influenced by questions from number theory.

From Dedekind's account of Dirichlet's lectures *Vorlesung über Zahlentheorie* (1863), Frobenius was also aware of the importance of characters of abelian groups.

These occur in the Dirichlet L-function:

$$L(s,\chi) := \sum \frac{1}{\chi(n)n^{-s}}$$

(Key to Dirchlet's theorem on primes in arithmetic progressions.)

Characters are also throughout Gauß' theory of quadratic reciprocity, composition of forms, ...

Bei dem Beweise des Satzes, dass jede lineare Function einer Variabeln unendlich viele Primzahlen darstellt, wenn ihre Coefficienten theilerfremde ganze Zahlen sind, benutzte Dirichlet zum ersten Male gewisse Systeme von Einheitswurzeln, die auch in der nahe verwandten Frage nach der Anzahl der Idealclassen in einem Kreiskörper auftreten (vergl. die Bemerkung von Dedekind in Dirichlet's Vorlesungen über Zahlentheorie, 4. Aufl. S. 625), sowie bei der Verallgemeinerung jenes Satzes auf quadratische Formen und in den Untersuchungen über deren Eintheilung in Geschlechter. Die charakteristische Eigenschaft dieser Ausdrücke besteht nach Dedekind darin, dass sie von einer variabeln positiven ganzen Zahl n abhängige Grössen $\chi(n)$ sind, die nur eine endliche Anzahl von Werthen haben und der Bedingung

$$\chi(m)\chi(n) = \chi(mn)$$

genügen. Wie er in rein abstracter Form ausführt, lassen sich den Elementen $A, B, C, \cdots$ jeder endlichen Gruppe $\mathfrak{H}$ vertauschbarer Elemente (Abel'schen Gruppe) solche Einheitswurzeln $\chi(A), \chi(B), \chi(C), \cdots$ zuordnen, welche die Gleichungen

$$\chi(A)\chi(B) = \chi(AB)$$

befriedigen, und die er nach dem Vorgange von Gauss die *Charaktere der Gruppe* nannte.

First paragraph of *Über Gruppencharaktere*.

Remarkably, when Frobenius begun his study of the character table *he didn't yet know the connection to representation theory*!

He was attempting to answer a question of Dedekind, which Dedekind had stumbled upon in 1880 (!).

> *Zahlkörper*, I. § 3, IV. § 2 und 3, Acta Math. Bd. 8 und 9).
>
> Im April dieses Jahres theilte mir Dedekind eine Aufgabe mit, auf die er im Jahre 1880 gekommen war, und die, weil sie sowohl der Gruppentheorie wie der Determinantentheorie angehöre, mich seiner Meinung nach wohl interessiren dürfte, während ihn selbst ein näheres Eingehen darauf zu weit von seinen arithmetischen Untersuchungen abziehen würde. Ihre Lösung, die ich nächstens mittheilen zu können hoffe, brachte mich auf eine Verallgemeinerung des Begriffs der Charaktere auf beliebige endliche Gruppen. Diesen Begriff will ich hier entwickeln in der Meinung, dass durch seine Einführung die Gruppentheorie eine wesentliche Förderung und Bereicherung erfahren dürfte. Ein besonderes Interesse gewinnt die Theorie der Charaktere noch durch ihre merkwürdigen Beziehungen zu der Theorie der aus mehreren Haupteinheiten gebildeten complexen Grössen.

Last paragraph of introduction to *Über Gruppencharaktere*.

Dedekind's observation was the following:

Take a finite group $G = \{g_1, g_2, g_3, \ldots, g_n\}$.

We will take $G = S_3$:

$$g_1 = id, g_2 = (12), g_3 = (23), g_4 = (123), g_5 = (321), g_6 = (13).$$

Write down the matrix $(g_i^{-1} h_j)_{i,j=1}^n$:

$$M := \begin{pmatrix} g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\ g_2 & g_1 & g_4 & g_3 & g_6 & g_5 \\ g_3 & g_5 & g_1 & g_6 & g_2 & g_4 \\ g_5 & g_3 & g_6 & g_1 & g_4 & g_2 \\ g_4 & g_6 & g_2 & g_5 & g_1 & g_3 \\ g_6 & g_4 & g_5 & g_2 & g_3 & g_1 \end{pmatrix}$$

Dedekind's observation was the following:

Take a finite group $G = \{g_1, g_2, g_3, \ldots, g_n\}$.

We will take $G = S_3$:

$$g_1 = id, g_2 = (12), g_3 = (23), g_4 = (123), g_5 = (321), g_6 = (13).$$

Now treat the elements as (commuting) variables:

$$M := \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_2 & x_1 & x_4 & x_3 & x_6 & x_5 \\ x_3 & x_5 & x_1 & x_6 & x_2 & x_4 \\ x_5 & x_3 & x_6 & x_1 & x_4 & x_2 \\ x_4 & x_6 & x_2 & x_5 & x_1 & x_3 \\ x_6 & x_4 & x_5 & x_2 & x_3 & x_1 \end{pmatrix}$$

Remarkably, the determinant factors:

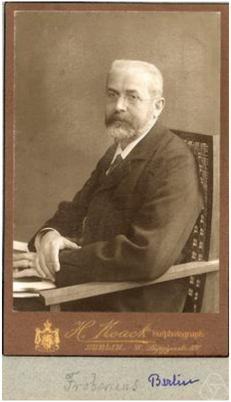
$$\det M = F_1 F_2 F_3^2$$

where:

$$\begin{aligned} F_1 &= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\ F_2 &= x_1 - x_2 - x_3 + x_4 + x_5 - x_6 \\ F_3 &= x_1^2 - x_1x_4 - x_1x_5 - x_2^2 + x_2x_3 + \\ &\qquad + x_2x_6 - x_3^2 + x_3x_6 + x_4^2 - x_4x_5 + x_5^2 - x_6^2 \end{aligned}$$

Dedekind asked whether a similar factorization held for any finite group.

Frobenius introduced characters, solved Dedekind's problem in general, and then realized the connection to linear representations the following year!



Georg Frobenius (1849 - 1917),
Born in Charlottenburg,
Professor in Berlin 1891-1917.

Williams Burnside (1852 - 1927).

In the hands of Frobenius and Burnside many beautiful results were discovered. The proofs were difficult to follow (despite Burnside's claims to the contrary).

Issai Schur, 1875 - 1941.

PhD in Berlin, 1901 under Frobenius,

Professor in Berlin 1909-1913, 1916-1934.

Emmy Noether, 1882 - 1935.

Weyl's obituary to Noether:

"a new and epoch-making style of thinking in algebra."

Weyl, *Emmy Noether*, Scripta Mathematica **3** (1935), 201-220.

First steps in modular representation theory: 1935 - 1960

We have so far discussed representations over $\mathbb{C}$.

The story remains the same over fields of characteristic not dividing $|G|$.

However over fields of small characteristic the situations becomes much more complicated.

Let $S_n \circlearrowright \mathbb{k}^n$ by permutation of the variables. (For $\mathbb{k}$ a field.)

Consider:

$$\Delta := \{(\lambda, \lambda, \dots, \lambda) \in \mathbb{k}^n \mid \lambda \in \mathbb{Z}\} \quad \text{"thin diagonal"}$$
$$\Sigma := \{(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{k}^n \mid \sum \lambda_i = 0\} \quad \text{"sum zero"}$$

Note $\sum_{i=1}^{n} \lambda = n\lambda$.

Hence $\Delta \cap \Sigma = 0$ if and only if $n \neq 0$ in k.

If $p \nmid n$, $\mathbb{k}^n = \Delta \oplus \Sigma$. ("complete reducibility")

If $p | n$, $\Delta \subset \Sigma \subset \mathbb{k}^n$.

In fact, in this case $\mathbb{k}^n$ is *indecomposable* as a representation of S_n.
("complete reducibility fails")

In fact, any representation of G over a field of characteristic p is completely reducible if and only if p does not divide $|G|$.

Why study modular representations?

1. Provides a way of recognising groups. (If I suspect that $G \cong SL_n(\mathbb{F}_q)$, I might like to proceed by constructing a representation of G on $\mathbb{F}_q^n$.)
2. Explains deep properties of the reduction modulo p of the character table.
3. Many representations occurring in (mathematical) nature are modular representations. (In number theory, algebraic geometry, ...)

Modular representation theory was developed almost single handedly by Richard Brauer (1901 - 1977) from 1935 - 1960.

Brauer was born in Berlin-Charlottenburg and wrote his thesis in Berlin under Issai Schur. He was forced to leave Germany in 1933 and wrote his first papers on modular representation theory in the period 1935 - 1940 in Toronto. Like Frobenius, Schur and Noether, number theory was an inspiration throughout his life.

Theorem (Brauer-Nesbitt)

Let $\mathbb{k}$ be an algebraically closed field of characteristic p. Then the number of irreducible representations of $\mathbb{k}G$ is equal to the number of p-regular conjugacy classes in G.

(A conjugacy class in G is *p-regular* if the order of any element is not divisible by p.)

The classification of finite simple groups: 1832 - 1981?

Since the beginning of group theory the simple groups have played an important role.

In 1832 Galois shows that $PSL_2(\mathbb{F}_q)$ is simple as long as $q \neq 2, 3$.

In Burnside's book in 1911 one sees a fascination in the possible orders and structures of simple groups.

Brauer's work on modular representation theory led to a good understanding of "small" simple groups.

The real breakthrough came in 1963.

NOTE M.

ON GROUPS OF ODD ORDER.

It has been seen that there is in some respects a marked difference between groups of even and those of odd order. The most noticeable property of groups of odd order is perhaps that they admit no self-inverse irreducible representation, except the identical one. From this property combined with that denoted by the relation

$$\Gamma^2 = \Gamma_{(2)} + \Sigma c_i \Gamma_i$$

of § 253, it is not difficult to shew that all irreducible groups of odd order in 3, 5 or 7 symbols are soluble.

Prof. G. A. Miller was the first to examine the possibility of a simple group of odd order under given conditions. In a paper in Vol. XXXIII (1901) of the *Proceedings of the London Mathematical Society* he proved that no group of odd order with a conjugate set of operations containing fewer than 50 members could be simple. In the same volume, working from a somewhat different point of view, the author proved that all transitive groups of odd order whose degree is less than 100 are soluble; and in his thesis (Baltimore, 1904) Mr H. L. Rietz extended this result to groups whose degrees are less than 243. The author has also shewn (*l.c.*) that the number of prime factors in the order of a simple group of odd order cannot be less than 7; and thence, by an examination of some particular cases, that 40,000 is a lower limit for the order of a group of odd degree, if simple. The contrast that these results shew between groups of odd and of even order suggests inevitably that simple groups of odd order do not exist. A discussion of the possibility of their existence must in any case lead to interesting results. Among other methods the problem might be approached by a detailed examination of the properties of irreducible groups of linear substitutions of odd order, or by regarding the group as a group of isomorphisms of an Abelian group of type (1, 1, ..., 1) whose order is a power of 2.

Burnside, *Theory of finite groups of finite order*, Second edition, 1911.

group of odd degree, if simple. The contrast that these results shew between groups of odd and of even order suggests inevitably that simple groups of odd order do not exist. A discussion of the

SOLVABILITY OF GROUPS OF ODD ORDER

WALTER FEIT AND JOHN G. THOMPSON

CHAPTER I

1. Introduction

The purpose of this paper is to prove the following result:

THEOREM. *All finite groups of odd order are solvable.*

Some consequences of this theorem and a discussion of the proof may be found in [11].

Feit, Thompson, *Solvability of groups of odd order*, Pacific J. Math, vol. 13, no. 3 (1963).

255 pages, one of the longest mathematical proofs at the time.

The Feit-Thompson theorem was the first evidence that a classification of simple groups might be possible. This is due to the Brauer-Fowler theorem (1955): there are only a finite number of simple groups with a give centralizer of an involution.

The rough idea is to study a potential simple group by studying the centralizer

$$C_G(\sigma) = \{g \in G \mid g\sigma = \sigma g\}$$

of an involution $\sigma \in G$ (σ exists by Feit-Thompson).

The central outstanding problem in the theory of finite groups today is that of determining the simple finite groups. One may say that this problem goes back to Galois. In any case, Camille Jordan must have been aware of it. Important classes

[. . . a discussion of John G. Thompson's work on the odd order theorem and finite simple groups . . .]

Let me finish with a personal remark. One reaches a point in life where one wonders what one still expects of life, what one would still like to see happen. This applies to events in Mathematics too. I have passed the point I mentioned. I like to say that I would like to see the solution of the problem of the finite simple groups and the part I expect Thompson's work to play in it. Quite generally, I would like to see to what further heights Thompson's future work will take him. I feel I should also say the same about the three other Fields medallists.

– Brauer, ICM address on the occasion of Thompson's Fields medal, 1970.

In 1983 the classification of finite simple groups was announced by Gorenstein. Since 2004 the experts agree:

Theorem

If G is a finite simple group then G is isomorphic to one of the following groups:

1. *a cyclic group of prime order;*
2. *an alternating group A_n for $n \geqslant 5$;*
3. *a finite group of Lie type ($PSL_n(\mathbb{F}_q), PSP_{2n}(\mathbb{F}_q), \ldots, E_8(\mathbb{F}_q)$)*
4. *one of the 26 sporadic simple groups.*

Wikipedia: "The proof of the classification theorem consists of tens of thousands of pages in several hundred journal articles."

The largest sporadic simple group is the monster simple group M.
A group of order:

$$8080, 17424, 79451,28758, 86459, 90496, 17107, 57005, 75436, 80000, 00000 =$$

$$= 2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.47.59.71$$

Conjectured to exist in early 70's by Fischer and Griess independently.

Character table calculated by Fischer, Livingstone and Thorne in 1979 (assuming it exists, and has an irreducible complex representation of dimension 196 883).

Proved to exist in 1982 by Griess.

$M = F_1$

	;	@	@	@	@	@	@	@	@	@
		808017424794512875886045	8309562962452852	139511839126	376561712757	14296150	272237831	33175342	26489012	4870492
		990496171075700575436800000000	823551610880000	3363281715200000	198516387840	077540249600	663616000	733090960000	826931200	9136640
		p power	A	A	A	A	A	B	A	B
		p' part	A	A	A	A	A	A	A	A
	ind	1A	2A	2B	3A	3B	3C	4A	4B	4C
χ_{1}	+	1	1	1	1	1	1	1	1	1
χ_{2}	+	196883	4371	275	782	53	-1	275	51	19
χ_{3}	+	21296876	91884	-2324	7889	-130	248	1772	-52	-20
χ_{4}	+	842609326	1139374	12974	55912	-221	-248	5878	782	-82
χ_{5}	+	18538750076	8507516	123004	249458	1598	248	28796	2652	380
χ_{6}	+	19360062527	9362495	-58305	297482	1508	-247	35903	-833	63
χ_{7}	+	293553734298	53981850	98970	1055310	-3927	3876	94874	1274	-102
χ_{8}	+	3879214937598	337044990	-690690	4751823	-4173	-3876	345598	-3874	-258
χ_{9}	+	36173193327999	1354188159	2864511	12616074	18954	0	701823	20383	-897
χ_{10}	+	125510727015275	3215883115	1219435	24688459	-25375	248	1223531	19499	-661
χ_{11}	+	190292345709543	2814161895	10249191	17144568	-8073	27000	734183	22151	-281
χ_{12}	+	222879856734249	3864186921	-7196631	26057022	11310	34503	1187881	-39831	1577
χ_{13}	+	1044868466775133	9223504989	-15756195	47292301	84448	31123	1762397	-46019	-2723
χ_{14}	+	1109944460516150	9697078070	26155830	40851749	97733	61256	1379126	87126	4662
χ_{15}	+	2374124840062976	22509162496	4100096	110509112	-1456	-61504	4100096	8192	4096
χ_{16}	o	8980616927734375	-2720265625	39414375	1603525	200200	-116375	-9625	-20825	8295
χ_{17}	o	8980616927734375	-2720265625	39414375	1603525	200200	-116375	-9625	-20825	8295
χ_{18}	+	15178147608537368	72990279960	-29873896	255751145	33254	-30628	7620888	-44200	-2280
χ_{19}	+	39660520552077425	128459630705	47061105	367316348	-227227	30380	9860529	82705	-3215
χ_{20}	+	60359800576579350	143552415510	71276310	371703759	-95718	120498	9004822	98294	-4074
χ_{21}	+	251098487132187500	336140827500	339995500	599239034	251810	-147250	11393900	512395	10860
χ_{22}	+	290568421805921077	444043629365	45093685	948362611	36400	-147497	20222773	-122059	4917
χ_{23}	+	336041615485626050	536115345090	-288233790	1087588073	-68263	61256	22648514	-153374	-2622
χ_{24}	+	2500435234254428856	1864421481144	319494840	2635262973	691236	116622	42900152	249016	-5448
χ_{25}	+	2986480825407204125	1612726090525	-385717475	1933299500	-57343	279495	28797725	-204867	7709
χ_{26}	o	3503434660075044981	-89143381899	-755269515	0	0	767637	184437	161109	39285
χ_{27}	o	3503434660075044981	-89143381899	-755269515	0	0	767637	184437	161109	39285
χ_{28}	+	3605718753596953125	2239938073125	284185125	2906660052	189540	0	43954725	345253	-20555
χ_{29}	+	8456836343580310400	3618137095040	753384320	3560377184	-1117324	-30628	46447488	903552	-13440
χ_{30}	+	8754193822112578125	3831845698125	-959677875	4307266557	-917865	27000	60123725	-883507	20045
χ_{31}	+	28585990950721640625	7669080680625	-804759375	6880196379	1056405	-813750	82024625	-832559	6065
χ_{32}	+	30815545786259524745	3936802874505	1656343689	2715826568	2163434	809627	25226377	437801	-63607
χ_{33}	+	31569817307122699605	8003575389525	2741767509	6723221931	-821106	-895869	76528981	1062517	-7083

$M = F_1$

	;	@	@	@	@	
		808017424794512875886459	8309562962452382	139511839126	376561712757	1429615
		904961710757005754368000000000	355161088000000	336328171520000	1985163878400	7754024960
		p power	A	A	A	
		p' part	A	A	A	
	ind	1A	2A	2B	3A	3
χ_1	+	1	1	1	1	
χ_2	+	196883	4371	275	782	9
χ_3	+	21296876	91884	-2324	7889	-1
χ_4	+	842609326	1139374	12974	55912	-22
χ_5	+	18538750076	8507516	123004	249458	159
χ_6	+	19360062527	9362495	-58305	297482	150
χ_7	+	293553734298	53981850	98970	1055310	-392
χ_8	+	3879214937598	337044990	-690690	4751823	-417

Modular representation theory and derived categories: 1980 - ???

Finally, I will turn to the present and the future.

Towards the end of Brauer's career, many researchers took up modular representation theory. Today it is a thriving area of pure mathematics.

There are two major (and related) new veins of investigation:

The study of *derived categories* in modular representation theory. Here the notion of derived equivalence plays a key role. Amazingly, derived categories often provide a means of explaining subtle and unexpected properties of character tables.

The study of *higher representation theory*. Here one considers the action of functors (like induction and restriction) on categories of reprentations and asks: What relations do these functors satisfy? What interesting structures can act on (higher) categories? This theory promises to become as powerful as classical representation theory.

Despite over a hundred years of effort we are still nowhere near answering the following question:

Question: What are the dimensions of the irreducible representations of the symmetric group S_n in characteristic p?

It seems to me that this question is similar to asking: what are the homotopy groups of spheres? It is so complicated we will never know the full story. But keeping it in mind and trying to solve it leads to deeper understanding and beautiful mathematics.

C12. The elementary divisors of $S^\lambda_{\mathbb{Z}}$ when $n = 12$.

λ	The dimension of D^λ				Elementary divisors
	$p=2$	$p=3$	$p=5$	$p=\infty$	
1^{12}	1	1	1	1	1^{1}
$2,1^{10}$	10	10	11	11	$1^{10}\cdot 12^{1}$
$2^2,1^8$	44	54	43	54	$1^{43}\cdot 5^{1}\cdot 10^{9}\cdot 110^{1}$
$2^3,1^6$	100	143	153	154	$1^{100}\cdot 4^{10}\cdot 8^{33}\cdot 24^{1}\cdot 72^{9}\cdot 360^{1}$
$2^4,1^4$	164	131	275	275	$1^{131}\cdot 3^{33}\cdot 6^{57}\cdot 42^{44}\cdot 168^{9}\cdot 504^{1}$
$2^5,1^2$	32	297	144	297	$1^{32}\cdot 2^{100}\cdot 4^{12}\cdot 20^{142}\cdot 140^{10}\cdot 280^{1}$
2^6	.	1	89	132	$2^{1}\cdot 6^{31}\cdot 12^{57}\cdot 60^{42}\cdot 420^{1}$
$3,1^9$	.	45	55	55	$2^{45}\cdot 24^{10}$
$3,2,1^7$	320	120	320	320	$1^{120}\cdot 3^{145}\cdot 9^{2}\cdot 99^{43}\cdot 297^{10}$
$3,2^2,1^5$	570	891	738	891	$1^{570}\cdot 2^{1}\cdot 14^{1}\cdot 28^{164}\cdot 56^{2}\cdot 280^{143}\cdot 1120^{10}$
$3,2^3,1^3$	1408	1013	372	1408	$1^{372}\cdot 5^{641}\cdot 15^{120}\cdot 45^{122}\cdot 225^{10}\cdot 675^{88}\cdot 4725^{55}$
$3,2^4,1$	288	10	835	1155	$1^{10}\cdot 3^{278}\cdot 6^{547}\cdot 30^{23}\cdot 120^{1}\cdot 240^{165}\cdot 720^{31}\cdot 1440^{100}$
$3^2,1^6$	.	210	573	616	$2^{210}\cdot 6^{141}\cdot 18^{120}\cdot 54^{101}\cdot 108^{1}\cdot 540^{43}$
$3^2,2,1^4$	1046	252	1925	1925	$1^{252}\cdot 3^{794}\cdot 6^{42}\cdot 24^{101}\cdot 48^{331}\cdot 144^{261}\cdot 432^{1}\cdot 1296^{143}$
$3^2,2^2,1^2$	416	2673	1957	2673	$1^{416}\cdot 2^{46}\cdot 4^{858}\cdot 28^{476}\cdot 56^{161}\cdot 280^{451}\cdot 1120^{101}\cdot 2240^{121}\cdot 11200^{43}$
$3^2,2^3$	.	45	1265	1320	$2^{45}\cdot 6^{262}\cdot 18^{109}\cdot 36^{334}\cdot 72^{284}\cdot 504^{231}\cdot 2520^{55}$
$3^3,1^3$	.	210	1650	1650	$2^{210}\cdot 6^{208}\cdot 12^{174}\cdot 36^{487}\cdot 252^{429}\cdot 504^{42}\cdot 2016^{100}$
$3^3,2,1$	1792	120	144	2112	$1^{120}\cdot 3^{24}\cdot 15^{442}\cdot 45^{513}\cdot 135^{693}\cdot 270^{99}\cdot 1890^{210}\cdot 9450^{11}$
3^4	.	.	89	462	$24^{89}\cdot 120^{31}\cdot 360^{211}\cdot 1080^{87}\cdot 2160^{43}\cdot 10800^{1}$
$4,1^8$	.	.	165	165	$6^{120}\cdot 72^{45}$
$4,2,1^6$	.	945	945	945	$2^{208}\cdot 4^{2}\cdot 8^{468}\cdot 88^{102}\cdot 176^{121}\cdot 704^{44}$
$4,2^2,1^4$	.	1431	1266	2376	$2^{1266}\cdot 10^{164}\cdot 20^{1}\cdot 60^{207}\cdot 300^{166}\cdot 600^{407}\cdot 4200^{165}$
$4,2^3,1^2$	.	1936	2135	3080	$2^{416}\cdot 4^{498}\cdot 8^{1022}\cdot 24^{11}\cdot 72^{120}\cdot 216^{68}\cdot 1080^{781}\cdot 2160^{164}$
$4,2^4$	.	54	1320	1485	$4^{54}\cdot 12^{362}\cdot 24^{210}\cdot 48^{1}\cdot 96^{288}\cdot 192^{405}\cdot 960^{34}\cdot 6720^{131}$
$4,3,1^5$	.	1728	1506	2079	$4^{252}\cdot 8^{1046}\cdot 16^{1}\cdot 32^{207}\cdot 160^{1}\cdot 320^{2}\cdot 640^{219}\cdot 1920^{351}$
$4,3,2,1^3$	5632	1428	1596	5632	$1^{1428}\cdot 3^{168}\cdot 15^{1825}\cdot 105^{299}\cdot 315^{341}\cdot 945^{513}\cdot 2835^{485}\cdot 14175^{573}$

Some dimensions of simple modules for $p = 2, 3, 5$ for S_{12}.

Mathas, *Iwahori-Hecke algebras and Schur algebras of the symmetric group.* University Lecture Series, 1999.

In 1990, following enormous calculations, Gordon James formulated a conjecture on the dimensions of the simple representations of S_n if $p > \sqrt{n}$ ("p not too small").

His conjecture if proven true, would represent major progress on the problem.

His conjecture is true for $n = 1, 2, \ldots, 22$.

James, *The decomposition matrices of $GL_n(q)$ for $n \leqslant 10$*, Proc. London Math. Soc. (3) 60 (1990), no. 2, 225–265.

The matrices Δ_{10} for $e = 3$

$n = 10,\ e = 3,\ p > 3$

(10)	1																														
(91)		1																													
(82)	1		1																												
(81^2)				1																											
(73)			1		1																										
(721)	1		1		1	1																									
(71^3)						1	1																								
(64)								1																							
(631)				1					1																						
(62^2)		1						1		1																					
(621^2)	1				1	1	1				1																				
(61^4)										1		1																			
(5^2)					1								1																		
(541)									1					1																	
(532)				1					1					1	1																
(531^2)																1															
(52^21)	1		1		1						1		1				1														
(521^3)							1				1						1	1													
(51^5)															1				1												
(4^22)								1		1										1											
(4^21^2)					1						1		1								1										
(43^2)					1	1							1									1									
(4321)	1		1		1	1	1				1		1				1				1	1	1								
(431^3)	1						1				1		1				1	1			1		1	1							
(42^3)	1		1														1						1		1						
(42^21^2)																										1					
(421^4)							1						1				1	1				1	1	1	1		1				
(41^6)																						1					1	1			
(3^31)	1					1	1															1	1						1		
(3^22^2)				1											1															1	
(3^221^2)		1								1		1								1											1
(3^21^4)														1	1				1											1	
(32^31)		1																													1
(32^21^3)												1								1											1
(321^5)							1						1								1	1	1	1	1		1	1	1		
(31^7)																				1											
(2^5)	1																						1	1	1				1		
(2^41^2)																														1	
(2^31^4)							1														1		1	1	1				1		
(2^21^6)													1								1			1				1			
(21^8)														1																	
(1^{10})													1											1							

Adjustment matrix

$n = 10 \quad (3^31) \quad 1$

Theorem (W, 2013)

The James conjecture fails "generically". In particular, it is not true for S_n for all $n \geqslant 1\,744\,860$.

The proof proceeds by constructing certain representations that are (much) smaller than the James conjecture predicts. It builds on earlier work of/with Soergel, Elias, Libedinsky and Xuhua He.

Theorem (W, 2013)

The James conjecture fails "generically". In particular, it is not true for S_n for all $n \geqslant 1\ 744\ 860$.

A key tool are techniques going back to Schur's PhD thesis in 1901 (one year after Frobenius first wrote down the character table of the symmetric group)!

We are trying to work out where, between $n = 22$ and $n = 1\ 744\ 860$, the conjecture first goes wrong. But it is not easy!

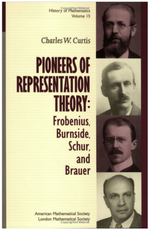
There is still much to say about S_n, possibly the most fundamental of all finite groups!

However in spite of all our efforts, we know very little about finite groups. The mystery has not been resolved, we cannot even say for sure whether order or chaos reigns. If any excitement can be derived from what I have to say, it should come from the feeling of being at a frontier across which we can see many landmarks, but which as a whole is unexplored, of planning ways to find out about the unknown, even if the pieces we can put together are few and far apart. My hope then is that some of you may go out with the idea: "Now let me think of something better myself."

– Richard Brauer, *On finite groups and their characters*,

Bull. Amer. Math. Soc. Volume 69, Number 2 (1963), 125-130.

Thanks!



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Curtis, *Pioneers of representation theory: Frobenius, Burnside, Schur, and Brauer.* History of Mathematics, 15. AMS, 1999.

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