Mathematics in light of representation theory

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BMS Friday Colloquium.

Symmetry and the notion of a group: 1832 - 1920.

Symmetry is all around us.

We have been observing and studying symmetry for millennia.

The set of symmetries form a group.

We owe the term group(e) to Galois (1832).



Galois theory:



Galois theory: This action tells us everything about f and its roots.

En d'autres termes, quand un groupe G en contient un autre H, le groupe G peut se partager en groupes, que l'on obtient chacun en opérant sur les permutations de H une même substitution ; en sorte que

 $G=H+HS+HS'+\ldots.$

 1. Écrite la veille de la mort de l'auteur. (Insérée en 1832 dans la Revue encyclopédique, numéro de septembre, page 568.)
 (J. LIOUVILLE.)

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Et aussi il peut se diviser en groupes qui ont tous les mêmes substitutions, en sorte que

 $G=H+TH+T^{\prime}H+\ldots.$

Ces deux genres de décompositions ne coïncident pas ordinairement. Quand ils coïncident, la décomposition est dite *propre*.

Il est aisé de voir que, quand le groupe d'une équation n'est susceptible d'aucune décomposition propre, on aura beau transformer cette équation, les groupes des équations transformées auront toujours le même nombre de permutations.

Au contraire, quand le groupe d'une équation est susceptible d'une décomposition propre, en sorte qu'il se partage en M groupes de N permutations, on pourra résoudre l'équation donnée au moyen de deux équations : l'une aura un groupe de M permutations, l'autre un de N permutations.

Lors donc qu'on aura épuisé sur le groupe d'une équation tout ce qu'il y a de décompositions propres possibles sur ce groupe, on arrivera à des groupes qu'on pourra transformer, mais dont les permutations seront toujours en même nombre.

Si ces groupes ont chacun un nombre premier de permutations, l'équation sera soluble par radicaux; sinon, non. $H \subset G$ is a subgroup

Letter to Auguste Chevalier in 1832

written on the eve of Galois' death

notion of a soluble group main theorem of Galois theory

notion of a normal subgroup

notion of a simple group

be noticed also, that if $\theta = \phi$, then, whatever the symbols α , β may be, $\alpha \theta \beta = \alpha \phi \beta$, and conversely.

A set of symbols,

1, α, β...

all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself belongs to the set, is said to be a group*. It follows that if the entire group is multiplied by any one of the symbols, either as further or nearer factor, the effect is simply to reproduce the group; or what is the same thing, that if the symbols of the group are multiplied together so as to form a table, thus :--

Further factors.

		1	α	β	••
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Nearer factors.	β	β	αβ	β^2	_
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that as well each line as each column of the square will contain all the symbols $1, \alpha, \beta$. It also follows that the product of any number of the symbols, with or without repetitions, and in any order whatever, is a symbol of the group. Suppose that the

Cayley, On the theory of groups, as depending on the symbolic equation $\Theta^n = 1$,

Philosophical Magazine, 4th series, 7, 1854.

Cayley (1878): "A group is defined by the law of composition of its members."

I. Je zwei Elemente A und B bestimmen in der angegebenen Reihenfolge eindeutig ein drittes, welches mit AB bezeichnet wird. $(g_1, g_2) \mapsto g_1g_2$.

II. Aus jeder der beiden Gleichungen AC = BC oder CA = CB folgt $gh = gh' \Rightarrow h = h'$ A = B. $gh = gh' \Rightarrow h = h'$

111. Für die Operation, durch welche AB aus A und B entspringt, gilt das associative Gesetz (AB) C = A(BC), aber nicht nothwendig das commutative Gesetz AB = BA.

IV. Die Anzahl der Elemente ist endlich. G is finite. \Rightarrow existence of inverses

- Frobenius, Neuer Beweis des Sylowschen Satzes, Crelle, 1884.

CHAPTER II.

THE DEFINITION OF A GROUP.

12. In the present chapter we shall enter on our main subject and we shall begin with definitions, explanations and examples of what is meant by a group.

Definition. Let

A, B, C,...

represent a set of operations, which can be performed on the same object or set of objects. Suppose this set of operations has the following characteristics.

(a) The operations of the set are all distinct, so that no two of them produce the same change in every possible application.

(β) The result of performing successively any number of operations of the set, say A, B, \ldots, K , is another definite operation of the set, which depends only on the component operations and the sequence in which they are carried out, and not on the way in which they may be regarded as associated. Thus A followed by B and B followed by C are operations of the set, say D and E; and D followed by C is the same operation as A followed by E.

(γ) A being any operation of the set, there is always another operation A_{-1} belonging to the set, such that A followed by A_{-1} produces no change in any object.

Burnside, Theory of groups of finite order, 1897. Surely the definition of group is one of the most intuitive and useful in all of mathematics.

Why did it take us so long to realize the importance of this notion?

Speculation: In Galois theory we first see the importance of the *structure of the symmetry*, or what we now call group theory.

That is, we move from the usefulness of *one* symmetry to the study of the *set of all* symmetries.

It was Galois who first asked:

Is a finite group G simple?

A group G is *simple* if it has no non-trivial normal subgroups.

A subgroup $H \subset G$ is normal if it is the kernel of some homomorphism $G \to G'$.

Jordan-Hölder theorem: finite simple groups are the building blocks of all finite groups.

We also see this is Klein's Erlangen program (1872): Geometry is its group of symmetries.

This idea also pervades 20th and 21st century theoretical physics.

Representation theory is the study of linear group actions:

A representation of a group G is a homomorphism

 $\rho: G \to GL(V)$

for some vector space V.

A representation is the same thing as a linear action of G on V.

A representation is *irreducible* if the only subspaces $U \subset V$ which are stable under the action of G are $\{0\} \subset V$ and V itself.

There is a Jordan-Hölder theorem: the irreducible representations are the building blocks of all representations.

A representation theorist's strategy:



Three examples of mathematics in light of representation theory

Example 1: Finite group actions on sets.

For a fixed finite group G these two problems are "the same": 1) classify finite sets with G-action; 2) classify subgroups $H \subset G$ up to conjugacy.

The equivalent problems turn out to be extremely complicated. Because every finite group is a subgroup of a symmetric group, a solution to (2) would be something like a classification of all finite groups. There are more than 30 papers on the classification of maximal subgroups of the monster simple group.

However the analogous linear problem "classify \mathbb{C} -vector spaces with linear *G*-action" is representation theory. Here we have a satisfactory answer for many groups.

Example 2: The circle and the Fourier transform.

Let
$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$
. Then S^1 is a (Lie) group.

For any $m \in \mathbb{Z}$ we have a one-dimensional representation of S^1 via:

$$S^1 \in z \mapsto z^m \in \mathbb{C}^* = GL_1(\mathbb{C}).$$

In fact, these are all irreducible representations of S^{1} !

Now we consider: $S^1 \subset S^1$.

We linearize this action and consider for example

 $S^1 \subset L^2(S^1, \mathbb{C}).$

Now our irreducible characters z^m belong to the right hand side.

Moreover, as Hilbert spaces:

$$L^2(S^1,\mathbb{C}) = \bigoplus \mathbb{C} z^m$$

If we identify $S^1 = \mathbb{R}/\mathbb{Z}$ then the functions z^m become the fundamental frequencies $\lambda \mapsto e^{2\pi i m \lambda}$ of Fourier analysis.

Moral: The decomposition of $L^2(S^1, \mathbb{C})$ into irreducible representations is the theory of Fourier series.

Similarly, the Fourier transform can be explained in terms of representations of $(\mathbb{R}, +)$, spherical harmonics in terms of representations of $SO(3) \subset S^2, \ldots$

Example 3: Rational points and Fermat's last theorem.

Suppose we want to find rational solutions to an equation X like:

$$y^2 = x^3 - x^2 - 24649x + 1355209$$

Let us write $X(\mathbb{C})$ for the solutions with $x, y \in \mathbb{C}$, $X(\mathbb{Q})$ for solutions $x, y \in \mathbb{Q}$ etc.

It turns out that $X(\mathbb{C})$ is a Riemann surface of genus one:



The points in an algebraic closure $X(\overline{\mathbb{Q}})$ are also "easy" (think of the stars in the night sky):



The tricky point is to find the rational points $X(\mathbb{Q})$:



Let $\operatorname{Gal}(\mathbb{Q})$ denote the absolute Galois group (automorphisms of $\mathbb{Q} \subset \overline{\mathbb{Q}}$). Group theory interpretation:



Diophantine geometry can be encoded in questions like: Understand the $Gal(\mathbb{Q})$ -action on $X(\overline{\mathbb{Q}})$. But we will probably never understand the $Gal(\mathbb{Q})$ sets $X(\overline{\mathbb{Q}})$. However representation theory suggests that we should cook up a linear object out of the action of $\operatorname{Gal}(\mathbb{Q})$ out of $X(\overline{\mathbb{Q}})$.

It turns out that we can do this, and it is *extremely* profitable. The short version: $\operatorname{Gal}(\mathbb{Q})$ acts in a very interesting way on $H_1(X; \mathbb{Q}_\ell) = \mathbb{Q}_\ell^2$. (Can be thought of as something like a tangent space.)



This is the structure behind the proof of Fermat's last theorem:

- 1. start with a solution $x^n + y^n = z^n$ with $x, y, z \in \mathbb{Z}$, n > 2;
- build from this solution a strange elliptic curve E (the "Frey curve");
- observe that such a curve would give a very strange G-representation H₁(E; ℚ₃) (Frey, Serre, Ribet);
- 4. show that such a *G*-representation cannot exist (Wiles, Taylor-Wiles).

Moreover the Langlands program gives us a vast array of theorems and conjectures linking representations of Galois groups coming from Diophantine problems (like the rational points question above) to analysis and automorphic forms.

A beautiful introduction to these ideas:

R. P. Langlands, *Representation theory: its rise and its role in number theory.* Proceedings of the Gibbs Symposium (New Haven, CT, 1989)

Representations of finite groups and the character table

Basic theorems in the representation theory of a finite group G:

1. any \mathbb{C} -representation of G is isomorphic to a direct sum of irreducible representations;

2.

$$\# \left\{ \begin{array}{c} \text{irreducible} \\ \mathbb{C}\text{-representations of } G \end{array} \right\}_{\cong} = \# \left\{ \begin{array}{c} \text{conjugacy} \\ \text{classes in } G \end{array} \right\}.$$

3. Any finite dimensional representation $\rho : G \rightarrow GL(V)$ is determined (up to isomorphism) by its *character*:

$$\chi_{\rho}: \mathcal{G} \to \mathbb{C}: \mathcal{g} \mapsto \operatorname{Tr} \rho(\mathcal{g}).$$

Hence, we know (almost) everything about the \mathbb{C} -representations of a group once we know the characters of the irreducible representations of our group G.

 $\chi(hgh^{-1}) = \mathsf{Tr}(\rho(hgh^{-1})) = \mathsf{Tr}(\rho(h)\rho(g)\rho(h)^{-1}) = \mathsf{Tr}(\rho(g)) = \chi(g).$

Hence χ is a function on the conjugacy classes of *G*.

All of this information can be conveniently displayed in the character table of G. The rows give the irreducible characters of G and the columns are indexed by the conjugacy classes of G.

The character table of G is the \mathbb{C} -linear shadow of G.

The first character table ever published. Here G is the alternating group on 4 letters, or equivalently the symmetries of the tetrahedron.

Frobenius, Über Gruppencharaktere, S'ber. Akad. Wiss. Berlin, 1896.

Now $G = S_5$, the symmetric group on 5 letters of order 120:

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	χ2	1	1	-1	2	-2	0	-1	10	
	Χ3	1	-1	-1	1	1	0	1	20	
	X4	1	-1	1	0	0	0	-1	30	
	X5	1	0	0	-1	-1	1	1	24	
	X6	1	1	-1	-1	1	0	-1	20	

Conway, Curtis, Norton, Parker, Wilson, Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With computational assistance from J. G. Thackray. Oxford University Press, 1985.

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But if you're not yet convinced you are not alone!

Cayley's dictum that "a group is defined by means of the laws of combination of its symbols" would imply that, in dealing purely with the theory of groups, no more concrete mode of representation should be used than is absolutely necessary. It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

- Burnside, *Theory of groups of finite order*, 1897. (One year after Frobenius' definition of the character.)

PREFACE TO THE SECOND EDITION

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is accordingly in the present edition a large amount of new matter.

Burnside, *Theory of groups of finite order*, Second edition, 1911.
 (15 years after Frobenius' definition of the character table.)

What led Frobenius to his marvellous definition?

As well as an inherent fascination in finite groups, Frobenius was influenced by questions from number theory.

From Dedekind's account of Dirichlet's lectures *Vorlesung über Zahlentheorie* (1863), Frobenius was also aware of the importance of characters of abelian groups.

These occur in the Dirichlet *L*-function:

$$L(s,\chi) := \sum \frac{1}{\chi(n)n^{-s}}$$

(Key to Dirchlet's theorem on primes in arithmetic progressions.)

Characters are also throughout Gauß' theory of quadratic reciprocity, composition of forms, . . .
Bei dem Beweise des Satzes, dass jede lineare Function einer Variabeln unendlich viele Primzahlen darstellt, wenn ihre Coefficienten theilerfremde ganze Zahlen sind, benutzte DIRICHLET zum ersten Male ^{8e}wisse Systeme von Einheitswurzeln, die auch in der nahe verwandten Frage nach der Anzahl der Idealclassen in einem Kreiskörper auftreten vergl. die Bemerkung von Dedekind in Dirichler's Vorlesungen über ^{Lahlentheorie,} 4. Aufl. S. 625), sowie bei der Verallgemeinerung jenes Satzes auf quadratische Formen und in den Untersuchungen über deren Eintheilung in Geschlechter. Die charakteristische Eigenschaft dieser Ausdrücke besteht nach Dedekind darin, dass sie von einer variabeln Positiven ganzen Zahl n abhängige Grössen $\chi(n)$ sind, die nur eine endliche Anzahl von Werthen haben und der Bedingung

 $\chi(m)\chi(n) = \chi(mn)$

^{gen}ügen. Wie er in rein abstracter Form ausführt, lassen sich den Elementen A, B, C, ... jeder endlichen Gruppe 5 vertauschbarer Ele-^{Inente} (ABEL'schen Gruppe) solche Einheitswurzeln $\chi(A), \chi(B), \chi(C), \dots$ zuordnen, welche die Gleichungen

$$\chi(A)\chi(B) = \dot{\chi}(AB)$$

befriedigen, und die er nach dem Vorgange von Gauss die Charaktere der Gruppe nannte.

First paragraph of Über Gruppencharaktere.

Remarkably, when Frobenius begun his study of the character table *he didn't yet know the connection to representation theory*!

He was attempting to answer a question of Dedekind, which Dedekind had stumbled upon in 1880 (!).

Example per, 1. § 3, 1V. § 2 und 3, Acta Math. Bd. 8 und 9). Im April dieses Jahres theilte mir DEDEKIND eine Aufgabe mit, auf die er im Jahre 1880 gekommen war, und die, weil sie sowohl der Gruppentheorie wie der Determinantentheorie angehöre, mich sei ner Meinung nach wohl interessiren dürfte, während ihn selbst ein näheres Eingehen darauf zu weit von seinen arithmetischen Unter suchungen abziehen würde. Ihre Lösung, die ich nächstens mittheilen zu können hoffe, brachte mich auf eine Verallgemeinerung des Begriffs der Charaktere auf beliebige endliche Gruppen. Diesen Begriff will ich hier entwickeln in der Meinung, dass durch seine Einführung die Gruppentheorie eine wesentliche Förderung und Bereicherung er fahren dürfte. Ein besonderes Interesse gewinnt die Theorie der Charaktere noch durch ihre merkwürdigen Beziehungen zu der Theorie der aus mehreren Haupteinheiten gebildeten complexen Grössen.

Last paragraph of introduction to Über Gruppencharaktere.

Dedekind's observation was the following:

Take a finite group $G = \{g_1, g_2, g_3, \dots, g_n\}.$ We will take $G = S_3$:

 $g_1 = id, g_2 = (12), g_3 = (23), g_4 = (123), g_5 = (321), g_6 = (13).$

Write down the matrix $(g_i^{-1}h_j)_{i,j=1}^n$:

$$M := \begin{pmatrix} g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\ g_2 & g_1 & g_4 & g_3 & g_6 & g_5 \\ g_3 & g_5 & g_1 & g_6 & g_2 & g_4 \\ g_5 & g_3 & g_6 & g_1 & g_4 & g_2 \\ g_4 & g_6 & g_2 & g_5 & g_1 & g_3 \\ g_6 & g_4 & g_5 & g_2 & g_3 & g_1 \end{pmatrix}$$

Dedekind's observation was the following:

Take a finite group
$$G = \{g_1, g_2, g_3, \dots, g_n\}$$
.
We will take $G = S_3$:

 $g_1 = id, g_2 = (12), g_3 = (23), g_4 = (123), g_5 = (321), g_6 = (13).$

Now treat the elements as (commuting) variables:

$$M := \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_2 & x_1 & x_4 & x_3 & x_6 & x_5 \\ x_3 & x_5 & x_1 & x_6 & x_2 & x_4 \\ x_5 & x_3 & x_6 & x_1 & x_4 & x_2 \\ x_4 & x_6 & x_2 & x_5 & x_1 & x_3 \\ x_6 & x_4 & x_5 & x_2 & x_3 & x_1 \end{pmatrix}$$

Remarkably, the determinant factors:

$$\det M = F_1 F_2 F_3^2$$

where:

$$F_{1} = x_{1} + x_{2} + x_{3} + x_{4} + x_{5} + x_{6}$$

$$F_{2} = x_{1} - x_{2} - x_{3} + x_{4} + x_{5} - x_{6}$$

$$F_{3} = x_{1}^{2} - x_{1}x_{4} - x_{1}x_{5} - x_{2}^{2} + x_{2}x_{3} + x_{2}x_{6} - x_{3}^{2} + x_{3}x_{6} + x_{4}^{2} - x_{4}x_{5} + x_{5}^{2} - x_{6}^{2}$$

Dedekind asked whether a similar factorization held for any finite group.

Frobenius introduced characters, solved Dedekind's problem in general, and then realized the connection to linear representations the following year!







Williams Burnside (1852 - 1927).

In the hands of Frobenius and Burnside many beautiful results were discovered. The proofs were difficult to follow (despite Burnside's claims to the contrary).



Issai Schur, 1875 - 1941.

PhD in Berlin, 1901 under Frobenius,

Professor in Berlin 1909-1913, 1916-1934.



Emmy Noether, 1882 - 1935.

Weyl's obituary to Noether:

"a new and epoch-making style of thinking in algebra."

Weyl, Emmy Noether, Scripta Mathematica 3 (1935), 201-220.

First steps in modular representation theory: 1935 - 1960

We have so far discussed representations over $\mathbb{C}.$

The story remains the same over fields of characteristic not dividing |G|.

However over fields of small characteristic the situations becomes much more complicated.

Let $S_n \subset \mathbb{k}^n$ by permutation of the variables. (For \mathbb{k} a field.) Consider:

$$\Delta := \{ (\lambda, \lambda, \dots, \lambda) \in \mathbb{k}^n \mid \lambda \in \mathbb{Z} \} \quad \text{``thin diagonal''}$$
$$\Sigma := \{ (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{k}^n \mid \sum \lambda_i = 0 \} \quad \text{``sum zero''}$$

Note
$$\sum_{i=1}^{n} \lambda = n\lambda$$
.

Hence $\Delta \cap \Sigma = 0$ if and only if $n \neq 0$ in k.

If $p \nmid n$, $\mathbb{k}^n = \Delta \oplus \Sigma$. ("complete reducibility")

If $p|n, \Delta \subset \Sigma \subset \mathbb{k}^n$.

In fact, in this case \mathbb{k}^n is *indecomposable* as a representation of S_n . ("complete reducibility fails") In fact, any representation of G over a field of characteristic p is completely reducible if and only if p does not divide |G|.

Why study modular representations?

- 1. Provides a way of recognising groups. (If I suspect that $G \cong SL_n(\mathbb{F}_q)$, I might like to proceed by constructing a representation of G on \mathbb{F}_q^n .)
- 2. Explains deep properties of the reduction modulo *p* of the character table.
- Many representations occurring in (mathematical) nature are modular representations. (In number theory, algebraic geometry, ...)

Modular representation theory was developed almost single handedly by Richard Brauer (1901 - 1977) from 1935 - 1960.



Brauer was born in Berlin-Charlottenburg and wrote his thesis in Berlin under Issai Schur. He was forced to leave Germany in 1933 and wrote his first papers on modular representation theory in the period 1935 - 1940 in Toronto. Like Frobenius, Schur and Noether, number theory was an inspiration throughout his life. Theorem (Brauer-Nesbitt)

Let \Bbbk be an algebraically closed field of characteristic p. Then the number of irreducible representations of $\Bbbk G$ is equal to the number of p-regular conjugacy classes in G.

(A conjugacy class in G is p-regular if the order of any element is not divisible by p.)

The classification of finite simple groups: 1832 - 1981?

Since the beginning of group theory the simple groups have played an important role.

In 1832 Galois shows that $PSL_2(\mathbb{F}_q)$ is simple as long as $q \neq 2, 3$.

In Burnside's book in 1911 one sees a fascination in the possible orders and structures of simple groups.

Brauer's work on modular representation theory led to a good understanding of "small" simple groups.

The real breakthrough came in 1963.

NOTE M.

ON GROUPS OF ODD ORDER.

It has been seen that there is in some respects a marked difference between groups of even and those of odd order. The most noticeable property of groups of odd order is perhaps that they admit no self-inverse irreducible representation, except the identical one. From this property combined with that denoted by the relation

$$\Gamma^2 = \Gamma_{(2)} + \Sigma c_i \Gamma_i$$

of § 253, it is not difficult to shew that all irreducible groups of odd order in 3, 5 or 7 symbols are soluble.

Prof. G. A. Miller was the first to examine the possibility of a simple group of odd order under given conditions. In a paper in Vol. XXXIII (1901) of the Proceedings of the London Mathematical Society he proved that no group of odd order with a conjugate set of operations containing fewer than 50 members could be simple. In the same volume, working from a somewhat different point of view, the author proved that all transitive groups of odd order whose degree is less than 100 are soluble; and in his thesis (Baltimore, 1904) Mr H. L. Rietz extended this result to groups whose degrees are less than 243. The author has also shewn (l.c.)that the number of prime factors in the order of a simple group of odd order cannot be less than 7; and thence, by an examination of some particular cases, that 40,000 is a lower limit for the order of a group of odd degree, if simple. The contrast that these results shew between groups of odd and of even order suggests inevitably that simple groups of odd order do not exist. A discussion of the possibility of their existence must in any case lead to interesting results. Among other methods the problem might be approached by a detailed examination of the properties of irreducible groups of linear substitutions of odd order, or by regarding the group as a group of isomorphisms of an Abelian group of type (1, 1, ..., 1) whose order is a power of 2.

Burnside, *Theory of finite groups of finite order*, Second edition, 1911.

group of odd degree, if simple. The contrast that these results shew between groups of odd and of even order suggests inevitably that simple groups of odd order do not exist. A discussion of the

SOLVABILITY OF GROUPS OF ODD ORDER

WALTER FEIT AND JOHN G. THOMPSON

CHAPTER I

1. Introduction

The purpose of this paper is to prove the following result:

THEOREM. All finite groups of odd order are solvable.

Some consequences of this theorem and a discussion of the proof may be found in [11].

Feit, Thompson, *Solvability of groups of odd order*, Pacific J. Math, vol. 13, no. 3 (1963).

255 pages, one of the longest mathematical proofs at the time.

The Feit-Thompson theorem was the first evidence that a classification of simple groups might be possible. This is due to the Brauer-Fowler theorem (1955): there are only a finite number of simple groups with a give centralizer of an involution.

The rough idea is to study a potential simple group by studying the centralizer

$$C_{G}(\sigma) = \{g \in G \mid g\sigma = \sigma g\}$$

of an involution $\sigma \in G$ (σ exists by Feit-Thompson).

The central outstanding problem in the theory of finite groups today is that of determining the simple finite groups. One may say that this problem goes back to Galois. In any case, Camille Jordan must have been aware of it. Important classes

[...a discussion of John G. Thompson's work on the odd order theorem and finite simple groups ...]

Let me finish with a personal remark. One reaches a point in life where one wonders what one still expects of life, what one would still like to see happen. This applies to events in Mathematics too. I have passed the point I mentioned. I like to say that I would like to see the solution of the problem of the finite simple groups and the part I expect Thompson's work to play in it. Quite generally, I would like to see to what further heights Thompson's future work will take him. I feel I should also say the same about the three other Fields medallists.

- Brauer, ICM address on the occasion of Thompson's Fields medal, 1970.

In 1983 the classification of finite simple groups was announced by Gorenstein. Since 2004 the experts agree:

Theorem

If G is a finite simple group then G is isomorphic to one of the following groups:

- 1. a cyclic group of prime order;
- 2. an alternating group A_n for $n \ge 5$;
- 3. a finite group of Lie type $(PSL_n(\mathbb{F}_q), PSP_{2n}(\mathbb{F}_q), \dots, E_8(\mathbb{F}_q))$
- 4. one of the 26 sporadic simple groups.

Wikipedia: "The proof of the classification theorem consists of tens of thousands of pages in several hundred journal articles."

The largest sporadic simple group is the monster simple group M. A group of order:

 $= 2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.47.59.71$

Conjectured to exist in early 70's by Fischer and Griess independently.

Character table calculated by Fischer, Livingstone and Thorne in 1979 (assuming it exists, and has an irreducible complex representation of dimension 196 883).

Proved to exist in 1982 by Griess.

 $M = F_1$

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Modular representation theory and derived categories: 1980 - ???

Finally, I will turn to the present and the future.

Towards the end of Brauer's career, many researchers took up modular representation theory. Today it is a thriving area of pure mathematics. There are two major (and related) new veins of investigation:

The study of *derived categories* in modular representation theory. Here the notion of derived equivalence plays a key role. Amazingly, derived categories often provide a means of explaining subtle and unexpected properties of character tables.

The study of *higher representation theory*. Here one considers the action of functors (like induction and restriction) on categories of reprentations and asks: What relations do these functors satisfy? What interesting structures can act on (higher) categories? This theory promises to become as powerful as classical representation theory.

Despite over a hundred years of effort we are still nowhere near answering the following question:

Question: What are the dimensions of the irreducible representations of the symmetric group S_n in characteristic p?

It seems to me that this question is similar to asking: what are the homotopy groups of spheres? It is so complicated we will never know the full story. But keeping it in mind and trying to solve it leads to deeper understanding and beautiful mathematics.

λ	Т	he dime	ension of	f D^{λ}	Elementary divisors
	p=2	p=3	p=5	$p = \infty$	
112	1	1	1	1	11
$2, 1^{10}$	10	10	11	11	$1^{10} \cdot 12^{1}$
$2^2, 1^8$	44	54	43	54	$1^{43} \cdot 5^1 \cdot 10^9 \cdot 110^1$
$2^3, 1^6$	100	143	153	154	$1^{100} \cdot 4^{10} \cdot 8^{33} \cdot 24^1 \cdot 72^9 \cdot 360^1$
$2^4, 1^4$	164	131	275	275	$1^{131} \cdot 3^{33} \cdot 6^{57} \cdot 42^{44} \cdot 168^9 \cdot 504^1$
$2^5, 1^2$	32	297	144	297	$1^{32} \cdot 2^{100} \cdot 4^{12} \cdot 20^{142} \cdot 140^{10} \cdot 280^{11}$
2^{6}		1	89	132	$2^1 \cdot 6^{31} \cdot 12^{57} \cdot 60^{42} \cdot 420^1$
$3, 1^9$		45	55	55	$2^{45} \cdot 24^{10}$
$3, 2, 1^7$	320	120	320	320	$1^{120} \cdot 3^{145} \cdot 9^2 \cdot 99^{43} \cdot 297^{10}$
$3, 2^2, 1^5$	570	891	738	891	$1^{570} \cdot 2^1 \cdot 14^1 \cdot 28^{164} \cdot 56^2 \cdot 280^{143} \cdot 1120^{10}$
$3, 2^3, 1^3$	1408	1013	372	1408	$1^{372} \cdot 5^{641} \cdot 15^{120} \cdot 45^{122} \cdot 225^{10} \cdot 675^{88} \cdot 4725^{55}$
$3, 2^4, 1$	288	10	835	1155	$1^{10} \cdot 3^{278} \cdot 6^{547} \cdot 30^{23} \cdot 120^1 \cdot 240^{165} \cdot 720^{31} \cdot 1440^{100}$
$3^2, 1^6$		210	573	616	$2^{210} \cdot 6^{141} \cdot 18^{120} \cdot 54^{101} \cdot 108^1 \cdot 540^{43}$
$3^2, 2, 1^4$	1046	252	1925	1925	$1^{252} \cdot 3^{794} \cdot 6^{42} \cdot 24^{101} \cdot 48^{331} \cdot 144^{261} \cdot 432^1 \cdot 1296^{143}$
$3^2, 2^2, 1^2$	416	2673	1957	2673	$1^{416} \cdot 2^{46} \cdot 4^{858} \cdot 28^{476} \cdot 56^{161} \cdot 280^{451} \cdot 1120^{101} \cdot 2240^{121} \cdot 11200^{43}$
$3^2, 2^3$		45	1265	1320	$2^{45} \cdot 6^{262} \cdot 18^{109} \cdot 36^{334} \cdot 72^{284} \cdot 504^{231} \cdot 2520^{55}$
$3^3, 1^3$		210	1650	1650	$2^{210} \cdot 6^{208} \cdot 12^{174} \cdot 36^{487} \cdot 252^{429} \cdot 504^{42} \cdot 2016^{100}$
$3^3, 2, 1$	1792	120	144	2112	$1^{120} \cdot 3^{24} \cdot 15^{442} \cdot 45^{513} \cdot 135^{693} \cdot 270^{99} \cdot 1890^{210} \cdot 9450^{11}$
34			89	462	$24^{89} \cdot 120^{31} \cdot 360^{211} \cdot 1080^{87} \cdot 2160^{43} \cdot 10800^{1}$
$4, 1^{8}$			165	165	$6^{120} \cdot 72^{45}$
$4, 2, 1^6$		945	945	945	$2^{208} \cdot 4^2 \cdot 8^{468} \cdot 88^{102} \cdot 176^{121} \cdot 704^{44}$
$4, 2^2, 1^4$		1431	1266	2376	$2^{1266} \cdot 10^{164} \cdot 20^1 \cdot 60^{207} \cdot 300^{166} \cdot 600^{407} \cdot 4200^{165}$
$4, 2^3, 1^2$		1936	2135	3080	$2^{416} \cdot 4^{498} \cdot 8^{1022} \cdot 24^{11} \cdot 72^{120} \cdot 216^{68} \cdot 1080^{781} \cdot 2160^{164}$
$4, 2^4$		54	1320	1485	$4^{54} \cdot 12^{362} \cdot 24^{210} \cdot 48^1 \cdot 96^{288} \cdot 192^{405} \cdot 960^{34} \cdot 6720^{131}$
$4, 3, 1^5$		1728	1506	2079	$4^{252} \cdot 8^{1046} \cdot 16^1 \cdot 32^{207} \cdot 160^1 \cdot 320^2 \cdot 640^{219} \cdot 1920^{351}$
$4, 3, 2, 1^3$	5632	1428	1596	5632	$1^{1428} \cdot 3^{168} \cdot 15^{1825} \cdot 105^{299} \cdot 315^{341} \cdot 945^{513} \cdot 2835^{485} \cdot 14175^{573}$

DIVISORS OF INTEGRAL

SPECHT MODULES

Some dimensions of simple modules for p = 2, 3, 5 for S_{12} .

Mathas, Iwahori-Hecke algebras and Schur algebras of the symmetric group. University Lecture Series, 1999.

In 1990, following enormous calculations, Gordon James formulated a conjecture on the dimensions of the simple representations of S_n if $p > \sqrt{n}$ ("p not too small").

His conjecture if proven true, would represent major progress on the problem.

His conjecture is true for n = 1, 2, ..., 22.

James, The decomposition matrices of $GL_n(q)$ for $n \leq 10$, Proc. London Math. Soc. (3) 60 (1990), no. 2, 225–265.

The matrices Δ_{10} for $e = 5$																													
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Adjustment matrix

The matrices Δ_{10} for e = 3

n = 10 (3³1) 1

Theorem (W, 2013)

The James conjecture fails "generically". In particular, it is not true for S_n for all $n \ge 1$ 744 860.

The proof proceeds by constructing certain representations that are (much) smaller than the James conjecture predicts. It builds on earlier work of/with Soergel, Elias, Libedinsky and Xuhua He.

Theorem (W, 2013)

The James conjecture fails "generically". In particular, it is not true for S_n for all $n \ge 1$ 744 860.

A key tool are techniques going back to Schur's PhD thesis in 1901 (one year after Frobenius first wrote down the character table of the symmetric group)!

We are trying to work out where, between n = 22 and n = 1 744 860, the conjecture first goes wrong. But it is not easy!

There is still much to say about S_n , possibly the most fundamental of all finite groups!

However in spite of all our efforts, we know very little about finite groups. The mystery has not been resolved, we cannot even say for sure whether order or chaos reigns. If any excitement can be derived from what I have to say, it should come from the feeling of being at a frontier across which we can see many landmarks, but which as a whole is unexplored, of planning ways to find out about the unknown, even if the pieces we can put together are few and far apart. My hope then is that some of you may go out with the idea: "Now let me think of something better myself."

- Richard Brauer, On finite groups and their characters,

Bull. Amer. Math. Soc. Volume 69, Number 2 (1963), 125-130.

Thanks!



Curtis, *Pioneers of representation theory: Frobenius, Burnside, Schur, and Brauer.* History of Mathematics, 15. AMS, 1999.



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