Exercises for Algebra II
Bonn, WS 2012/13

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Revision Sheet 13

Hand in on Thursday, 31th January, prior to the lecture.

Exercise 1

Let V be a finite-dimensional vector space over a field k and $\mathfrak{g}$ a Lie subalgebra of $\mathfrak{gl}(V)$.

(a) Assuming that $\mathfrak{g}$ consists of nilpotent endomorphisms, show that $\mathfrak{g}$ is nilpotent (as a Lie algebra).

(b) Prove or disprove the converse to (a).

Exercise 2

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over some field. Show that $\dim(\mathfrak{g}/\mathfrak{z}(\mathfrak{g})) \neq 1$.

Exercise 3

Let $\mathfrak{g}$ be a Lie algebra and V, W two $\mathfrak{g}$-modules over some base field k.

(a) Show that $X.(v \otimes w) := X.v \otimes w + v \otimes X.w$ turns $V \otimes_k W$ into a $\mathfrak{g}$-module.

(b) Show that $(X.\varphi)(v) := -\varphi(X.v)$ turns $V^* := \mathrm{Hom}_k(V, k)$ into a $\mathfrak{g}$-module.

Remember that, up to isomorphism, for any $n \in \mathbb{N}_0$ there is a unique $n+1$-dimensional simple representation of $\mathfrak{sl}_2(\mathbb{C})$ and that these modules constitute all finite-dimensional simple representations of $\mathfrak{sl}_2(\mathbb{C})$. In this exercise, we denote this $n+1$-dimensional simple module by $L(n)$. By Weyl's theorem, any finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module isomorphic to a sum of these $L(n)$.

(a) Given $n, m \in \mathbb{N}_0$, provide a decomposition of the tensor product $L(n) \otimes_{\mathbb{C}} L(m)$ into a sum of irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules. (*Hint: Try to use only the knowledge of the weight spaces of $L(n) \otimes_{\mathbb{C}} L(m)$! It might also by helpful to consider some examples first.*)

(b) Show that $L(n)^* \cong L(n)$ as $\mathfrak{sl}_2(\mathbb{C})$-modules.

Exercise 4

Let $\mathfrak{g}$ be a simple, finite-dimensional, complex Lie algebra and $\kappa : \mathfrak{g} \times \mathfrak{g} \to k$ be a bilinear form on $\mathfrak{g}$ which is *invariant*, i.e. $\kappa([X, Y], Z) = \kappa(X, [Y, Z])$ for all $X, Y, Z \in \mathfrak{g}$.

(a) Show that $\mathfrak{g} \to \mathfrak{g}^*$, $X \mapsto \kappa(X, -)$ is a homomorphism of $\mathfrak{g}$-modules.

(b) Considering also $\mathfrak{g} \to \mathfrak{g}^*$, $X \mapsto \kappa_{\mathfrak{g}}(X, -)$, use Schur's lemma to show that κ is a scalar multiple of the Killing form $\kappa_{\mathfrak{g}}$.

Exercise 5

Recall that $\mathfrak{so}_{2n+1}(\mathbb{C}) = \{A \in \mathfrak{gl}_{2n+1}(\mathbb{C}) \mid A^t J + JA = 0\}$, where $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_n \\ 0 & 1_n & 0 \end{pmatrix}$.

(a) Check that $\mathfrak{so}_{2n+1}(\mathbb{C}) = \left\{ \begin{pmatrix} 0 & a & b \\ -b^t & A & B \\ -a^t & C & -A^t \end{pmatrix} \right\}$.

(b) Give an explicit description of the root space decomposition of $\mathfrak{so}_{2n+1}(\mathbb{C})$ with respect to the Cartan subalgebra of diagonal matrices.

Exercise 6

(a) Let $G \subseteq \mathrm{GL}(V)$ and $H \subseteq \mathrm{GL}(W)$ be linear Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and let $\rho : G \to H$ be a homomorphism of Lie groups. Further, let $H' \subseteq H$ be another linear Lie group in $\mathrm{GL}(W)$ with Lie algebra $\mathfrak{h}' \subseteq \mathfrak{h}$.

Show that $G' := \rho^{-1}(H')$ is linear Lie group in $\mathrm{GL}(V)$ with $\mathrm{Lie}(G') = \mathrm{Lie}(\rho)^{-1}(\mathfrak{h}')$.

(b) For a finite-dimensional real vector space V and $v \in V$, we put $\mathrm{Stab}(v) := \{\varphi \in \mathrm{GL}(V) \mid \varphi(v) = v\}$. Show that $\mathrm{Stab}(v)$ is a linear Lie group in $\mathrm{GL}(V)$, and that $\mathrm{Lie}(\mathrm{Stab}(v)) = \mathrm{stab}(v) := \{\psi \in \mathrm{End}(V) \mid \psi(v) = 0\}$.

Exercise 7

Let V be a finite-dimensional real vector space and $R \subset V$ be a reduced root system with Weyl group $W := W(R)$. Further, let Δ be a basis of R. Suppose that $\alpha, \alpha_1, ..., \alpha_n \in \Delta$ and that $w = s_{\alpha_1} ... s_{\alpha_n} \in W$ satisfies $w(\alpha) < 0$ (with respect to Δ).

(a) Show that there exists some $1 \leq i \leq n$ such that $s_{\alpha_{i+1}} ... s_{\alpha_n}(\alpha) = \alpha_i$.

(b) For i as in (a), show that $ws_\alpha = s_{\alpha_1} ... s_{\alpha_{i-1}} s_{\alpha_{i+1}} ... s_{\alpha_n}$.

Exercise 8

Let V be a finite-dimensional complex vector space and $\rho : \mathrm{SU}(2;\mathbb{C}) \to \mathrm{GL}_{\mathbb{C}}(V)$ be a complex representation of $\mathrm{SU}(2;\mathbb{C})$. For $n \in \mathbb{Z}$, we put

$$V_n = \left\{ v \in V \;\middle|\; \forall \lambda \in S^1 : \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} .v = \lambda^n v \right\}$$

(a) Let $\mathrm{Lie}(\rho) : \mathfrak{su}_2(\mathbb{C}) \to \mathfrak{gl}_{\mathbb{C}}(V)$ be the induced representation of $\mathfrak{su}_2(\mathbb{C})$. Show that

$$V_n = \left\{ v \in V \;\middle|\; \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} .v = inv \right\}.$$

(b) Use your knowledge of $\mathfrak{sl}_2(\mathbb{C})$-modules to show that $V = \bigoplus_{n \in \mathbb{Z}} V_n$.