Exercises for Algebra II
Bonn, WS 2012/13

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Sheet 11

Hand in on Thursday, 17th January, prior to the lecture.

Exercise 1

Let V be a finite-dimensional real vector space. Further, let $\{G_i\}_{i\in I}$ be an arbitrary family of linear Lie groups in $\mathrm{GL}(V)$.

(a) Show that $\bigcap\limits_{i\in I} G_i$ is a linear Lie group in $\mathrm{GL}(V)$. (2 points)

(b) Show that $\mathrm{Lie}\left(\bigcap\limits_{i\in I} G_i\right) = \bigcap\limits_{i\in I} \mathrm{Lie}(G_i)$. (2 points)

Exercise 2

Let V be a finite-dimensional real vector space and let A be a real subalgebra of $\mathrm{End}(V)$, i.e. A is a real subvector space of $\mathrm{End}(V)$ with $\mathrm{id}_V \in A$ such that for all $\psi, \phi \in A$ we have $\psi\phi \in A$, too.

Put $A^\times := \{\psi \in A \mid \exists \phi \in A : \psi\phi = \mathrm{id}_V\}$, the units in A.

(a) Show that $A^\times = A \cap \mathrm{GL}(V)$. (*Hint: Cayley-Hamilton*) (2 points)

(b) Show that $A^\times$ is a linear Lie group in $\mathrm{GL}(V)$. (1 point)

(c) Show that $\mathrm{Lie}(A^\times) = A$. (1 point)

As an application, consider a fixed $\phi \in \mathrm{End}(V)$ and put $C_\phi := \{\varphi \in \mathrm{GL}(V) \mid \varphi\phi = \phi\varphi\}$ as in the lecture.

(d) Conclude from (a)-(c) that $\mathrm{Lie}(C_\phi) = \{\psi \in \mathrm{End}(V) \mid \psi\phi = \phi\psi\}$. (1 point)

Specializing further, suppose V carries the structure of a complex vector space.

(e) Deduce from (d) that $\mathrm{Lie}(\mathrm{GL}_{\mathbb{C}}(V)) = \mathrm{End}_{\mathbb{C}}(V)$. (1 point)

Exercise 3

Let V be a finite-dimensional complex vector space with Hermitean inner product $\langle -, - \rangle$. Recall from the lecture that

$$U(V, \langle -, - \rangle) \ := \ \{\varphi \in \mathrm{GL}_{\mathbb{C}}(V) \ \mid \forall v, w \in V : \ \langle \varphi(v), \varphi(w) \rangle = \langle v, w \rangle\}$$

Determine $\mathrm{Lie}(U(V, \langle -, - \rangle))$. (*Hint: Use Exercises 1 and 2!*) (4 points)

Exercise 4

Let V be a finite-dimensional real vector space and $\mu : V \times V \to V$ be a bilinear map. As in the lecture, we define

$$\begin{aligned}
\mathrm{Aut}(V,\mu) &:= \{\ \varphi \in \mathrm{GL}(V) \ \mid\ \forall x,y \in V:\ \mu(\varphi(x),\varphi(y)) = \varphi(\mu(x,y))\} \\
\mathrm{Der}(V,\mu) &:= \{\ D \in \mathrm{End}(V) \ \mid\ \forall x,y \in V:\ D(\mu(x,y)) = \mu(D(x),y) + \mu(x,D(y))\}
\end{aligned}$$

(a) Show that $\mathrm{Aut}(V,\mu)$ is a linear Lie group. (1 point)

(b) Show that $\mathrm{Lie}(\mathrm{Aut}(V,\mu)) = \mathrm{Der}(V,\mu)$. (3 points)

In particular, if $V = \mathfrak{g}$ is a real Lie algebra with bracket $\mu = [-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, we get that $\mathrm{Aut}(\mathfrak{g})$ is a linear Lie group with $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})) = \mathrm{Der}(\mathfrak{g})$.