Exercises for Algebra II
Bonn, WS 2012/13

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Sheet 10

Hand in on Thursday, 10th January, prior to the lecture.

Exercise 1

Let G be a topological group and (V, ρ) be a continuous G-representation in a finite-dimensional complex vector space V (equipped with its unique norm topology). Further, let $U \subseteq V$ be a subrepresentation of V. Show that there exists a unique $\overline{\rho} : G \times V/U \to V/U$ such that

(a) $(V/U, \overline{\rho})$ is a G-representation, and

(b) The canonical projection $\pi : V \to V/U$ is G-linear, i.e.

$$\begin{tikzcd} G \times V \arrow[r, "\rho"] \arrow[d, "\mathrm{id}_G \times \pi"'] & V \arrow[d, "\pi"] \\ G \times V/U \arrow[r, "\overline{\rho}"'] & V/U \end{tikzcd}$$

is commutative.

The representation $(V/U, \rho)$ is called the *quotient of* V *by* U. (4 points)

Exercise 2

Let G be a compact topological group.

(a) Determine all compact subgroups of $(\mathbb{R}, +)$. (2 points)

(b) Show that any continuous group homomorphism $G \to \mathbb{C}^\times$ has image in $\mathbb{S}^1$.(2 points)

Exercise 3

Let G be an abelian topological group. Show that any simple, finite-dimensional and continuous representation of G is 1-dimensional, hence is given by some continuous group homomorphism $G \to \mathbb{C}^\times$ (or even $G \to \mathbb{S}^1$ if G is compact). (4 points)

Exercise 4

Let $G := (\mathbb{C}, +)$ and $\rho : G \to \mathrm{GL}_2(\mathbb{C})$ be given by $x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Show that ρ is not completely reducible. (4 points)

Frohe Weihnachten!