Dr. Geordie Williamson
Hanno Becker

Sheet 9

Hand in on Thursday, 13th December, prior to the lecture.

Exercise 1

Let $\phi : \mathfrak{g} \to \mathfrak{g}$ be an automorphism of a complex semi-simple Lie algebra $\mathfrak{g}$. Further, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and put $\mathfrak{h}' := \phi(\mathfrak{h})$.

(a) Show that $\mathfrak{h}'$ is also a Cartan subalgebra of $\mathfrak{g}$. (2 points)

(b) Show that ϕ induces an isomorphism of root systems $R(\mathfrak{g}, \mathfrak{h}) \cong R(\mathfrak{g}, \mathfrak{h}')$. (2 points)

Exercise 2

Recall that for semi-simple $\mathfrak{g}$ we denote $\mathrm{Inn}(\mathfrak{g})$ the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by the automorphisms $\exp(\mathrm{ad}(x))$ for x an ad-nilpotent element of $\mathfrak{g}$. Here we study the $\mathfrak{sl}_2$-case.

(a) Describe $\mathrm{Inn}(\mathfrak{sl}_2(\mathbb{C}))$. (*Hint:* Recall that $\exp(\mathrm{ad}(x))(y) = e^x y e^{-x}$) (2 points)

(b) Describe all Cartan subalgebras of $\mathfrak{sl}_2(\mathbb{C})$ and conclude that $\mathrm{Inn}(\mathfrak{g})$ acts transitively on the set of Cartan subalgebras of $\mathfrak{sl}_2(\mathbb{C})$. (2 points)

(c) Show that the transpose automorphism $A \mapsto -A^t$ of $\mathfrak{sl}_2(\mathbb{C})$ does not belong to $\mathrm{Inn}(\mathfrak{sl}_2(\mathbb{C}))$. (2 points)

Exercise 3

Let $\mathfrak{g}$, $\mathfrak{g}'$ be complex semi-simple Lie algebras with Cartan subalgebras $\mathfrak{h}$, $\mathfrak{h}'$ and simple systems $\Delta \subset R(\mathfrak{g}, \mathfrak{h})$, $\Delta' \subset R(\mathfrak{g}', \mathfrak{h}')$, respectively. Further, let $\Gamma(\Delta), \Gamma'(\Delta')$ be the corresponding Dynkin diagrams.

(a) Use Serre's Theorem to construct for any embedding $\Gamma(\Delta) \hookrightarrow \Gamma'(\Delta')$ a canonical embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}'$. (2 points)

In particular, any automorphism of $\Gamma(\Delta)$ induces a canonical automorphism of $\mathfrak{g}$.

(b) Take $\mathfrak{g}$ to be $\mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{h}$ resp. Δ to be the standard Cartan resp. the standard simple system. Give an explicit description of the automorphism of $\mathfrak{sl}_n(\mathbb{C})$ corresponding to the nontrivial automorphism of A_n. (2 points)

(c) In case $n = 2$, use Exercise 2 to check that the automorphism of part (b) differs from the transpose automorphism $A \mapsto -A^t$ of $\mathfrak{sl}_2(\mathbb{C})$ by an element of $\mathrm{Inn}(\mathfrak{sl}_2(\mathbb{C}))$, hence is not inner. (2 points)

Exercise 4

We finish the G_2-exercise from Sheet 8. To begin, here's a summary of the previous results, so that you can attack this second part even if you didn't do the first: We considered the complex 8×8-matrices

$$X_a := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_b := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$H_a := \mathrm{Diag}(1, -1, 2, 0, 0, -2, 1, -1), \quad H_b := \mathrm{Diag}(0, 1, -1, 0, 0, 1, -1, 0),$$

put $\mathfrak{h} := \mathbb{C}H_a + \mathbb{C}H_b$ and take as $\mathfrak{n}_+$ the subalgebra of $\mathfrak{gl}_8(\mathbb{C})$ generated by X_a and X_b. Here's what we already know:

- Defining $\alpha, \beta \in \mathfrak{h}^*$ by $\alpha(H_a) = 2, \alpha(H_b) = -1$ and $\beta(H_a) = -3, \beta(H_b) = 2$, X_a and X_b are weight vectors for the adjoint action of $\mathfrak{h}$ with weights α and β, respectively (i.e. $[H, X_a] = \alpha(H)X_a$ and $[H, X_b] = \beta(H)X_b$ for all $H \in \mathfrak{h}$).
- Together with X_a and X_b, the matrices $X_{a+b} := [X_a, X_b]$, $X_{2a+b} := \frac{1}{2}[X_a, X_{a+b}]$, $X_{3a+b} := \frac{1}{3}[X_a, X_{2a+b}]$ and $X_{3a+2b} := [X_b, X_{3a+b}]$ form a basis of $\mathfrak{n}_+$, and the $\mathfrak{h}$-weight of X_{ia+jb} is $i\alpha + j\beta$. These matrices are explicitly given by

$$X_{a+b} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{2a+b} = \begin{pmatrix} 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_{3a+b} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{3a+2b} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, to complete the construction, we take as $\mathfrak{n}_-$ the subalgebra of $\mathfrak{gl}_8(\mathbb{C})$ generated by X_a^t and X_b^t (where $(-)^t$ denotes the transpose) and put $\mathfrak{g} := \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$.

(a) Show that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}_8(\mathbb{C})$. (2 points)

(b) Provide an explicit decomposition of $\mathfrak{g}$ into $\mathfrak{h}$-eigenspaces. (1 point)

(c) Show that $\mathfrak{g}$ is simple and that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. (2 points)

(d) Conclude that $\mathfrak{g}$ is a simple Lie algebra of type G_2. (1 point)

Congratulations: you have constructed an exceptional simple Lie algebra! ☺