Exercises for Algebra II
Bonn, WS 2012/13

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Sheet 8

Hand in on Thursday, 6th December, prior to the lecture.

Exercise 1

Let $\mathfrak{g}$ be a semisimple complex Lie-algebra. Give a formula for the dimension of $\mathfrak{g}$ in terms of its root system! (4 points)

Exercise 2

In this exercise you are asked to determine a root space decomposition for the Lie algebra $\mathfrak{so}_{2n}(\mathbb{C})$. Recall that $\mathfrak{so}_{2n}(\mathbb{C})$ can be described as

$$\mathfrak{so}_{2n}(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathrm{Mat}_{n\times n}(\mathbb{C}), a = -d^t, b = -b^t, c = -c^t \right\}.$$

(a) Show that $\mathfrak{h} := \left\{ \begin{pmatrix} a & 0 \\ 0 & -a^t \end{pmatrix} \mid a \in \mathrm{Diag}_{n\times n}(\mathbb{C}) \right\}$ is a Cartan subalgebra of $\mathfrak{so}_{2n}(\mathbb{C})$ and determine the corresponding root space decomposition. (4 points)

(b) Determine the type of the root system of $\mathfrak{so}_{2n}(\mathbb{C})$. (1 point)

(c) Check your dimension formula from Exercise 1 for $\mathfrak{so}_{2n}(\mathbb{C})$! (1 point)

Exercise 3

Here you are asked to work out a root space decomposition for $\mathfrak{sp}_{2n}(\mathbb{C})$. Recall

$$\mathfrak{sp}_{2n}(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathrm{Mat}_{n\times n}(\mathbb{C}), a = -d^t, b = b^t, c = c^t \right\}.$$

Now, just as in Exercise 2, do the following:

(a) Show that $\mathfrak{h} := \left\{ \begin{pmatrix} a & 0 \\ 0 & -a^t \end{pmatrix} \mid a \in \mathrm{Diag}_{n\times n}(\mathbb{C}) \right\}$ is a Cartan subalgebra of $\mathfrak{sp}_{2n}(\mathbb{C})$ and determine the corresponding root space decomposition. (4 points)

(b) Determine the type of the root system of $\mathfrak{sp}_{2n}(\mathbb{C})$. (1 point)

(c) Check your dimension formula from Exercise 1 for $\mathfrak{sp}_{2n}(\mathbb{C})$! (1 point)

Exercise 4

This exercise describes the first and essential part of a construction of the simple Lie algebra of type G_2 and will be continued on the next sheet. Consider the following complex 8×8-matrices:

$$X_a := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_b := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$H_a := \mathrm{Diag}(1,-1,2,0,0,-2,1,-1), \quad H_b := \mathrm{Diag}(0,1,-1,0,0,1,-1,0).$$

Put $\mathfrak{h} := \mathbb{C}H_a + \mathbb{C}H_b$ and let $\mathfrak{n}$ be the subalgebra of $\mathfrak{gl}_8(\mathbb{C})$ generated by X_a and X_b.

(a) Show that X_a and X_b are eigenvectors for the adjoint action of $\mathfrak{h}$, and determine their weights. (2 points)

(b) Show that $\mathfrak{n}$ is 6-dimensional and determine the set $\Phi \subset \mathfrak{h}^*$ of weights of $\mathfrak{n}$ with respect to $\mathfrak{h}$. (4 points)

(c) Show that $\Phi \cup -\Phi \subseteq \mathfrak{h}^*$ is a root system of type G_2. (2 points)

Hint for b): In the end we want $\mathfrak{n}$ to be the positive part of a simple Lie algebra of type G_2, with X_a and X_b spanning the root spaces for the simple roots, and H_a, H_b being the corresponding coroots. Now, knowing the Cartan matrix, you can determine all positive roots (see Exercise 3b from Sheet 5), and it has to be possible to get basis vectors for the corresponding root spaces by taking iterated brackets of X_a and X_b, since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ for every semisimple Lie algebra $\mathfrak{g}$ and roots α, β. Use this background knowledge to write down a subspace of $\mathfrak{g}$ of dimension 6, and check by hand afterwards that it is indeed a Lie subalgebra of $\mathfrak{gl}_8(\mathbb{C})$.