Exercises for Algebra II — Dr. Geordie Williamson
Bonn, WS 2012/13 — Hanno Becker

Sheet 7

Hand in on Thursday, 29th November, prior to the lecture.

Exercise 1

A *free unital commutative k-algebra* on I is a pair (A, f) consisting of a unital, commutative k-algebra A together with a map of sets $f : I \to A$ such that any map of sets $g : I \to B$ with B another unital, commutative k-algebra factors uniquely through a homomorphism of unital k-algebras $A \to B$.

Show that the polynomial ring $k[X_i \mid i \in I]$ together with the map $i \mapsto X_i$ is a free unital commutative k-algebra on I. (4 points)

Exercise 2

Fix a field k and a set I. A *free unital k-algebra* on I is defined as in Exercise 1, removing the commutativity assumption everywhere.

Let $(F(I), I \to F(I))$ be a free Lie-algebra on I. Show that $\mathcal{U}(F(I))$ together with the composition $I \to F(I) \to \mathcal{U}(F(I))$ is a free unital k-algebra on I. (4 points)

(*If you know about adjoint functors: what's the general statement behind here?*)

Exercise 3

Let $\mathfrak{g}$ be a Lie algebra over a field. Show that the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ has no zero-divisors. (*Hint: Use the Poincaré-Birkhoff-Witt-Theorem*) (4 points)

Exercise 4

Let $\mathfrak{g}$ be a Lie algebra over a field k.

(a) Let $D : \mathfrak{g} \to \mathfrak{g}$ be a derivation of $\mathfrak{g}$. Show that there exists a unique derivation $\widetilde{D} : \mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{g}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "D"] \arrow[d, "\iota"'] & \mathfrak{g} \arrow[d, "\iota"] \\ \mathcal{U}\mathfrak{g} \arrow[r, "\widetilde{D}"'] & \mathcal{U}\mathfrak{g} \end{tikzcd}$$

(*Hint:* Express derivations on an algebra A in terms of special algebra homomorphisms to the matrix algebra $\begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$.) (3 points)

(b) Show that $\widetilde{\mathrm{ad}(x)} = [\iota(x), -]$ for all $x \in \mathfrak{g}$. (1 point)

$*$-Exercise

In this last exercise we construct a basis for the free Lie algebra $F(I)$ on a set I, where we realize $F(I)$ concretely inside the k-algebra $T(I)$ of formal k-linear combinations of words in I (a concrete model for the free unital k-algebra on I) as the Lie subalgebra generated by elements of I (considered as one-element words).
Equip I with a linear order $<$, and also denote by $<$ the induced lexicographic order on the set I^* of words in I. A non-empty word $w \in I^*$ is called a *Lyndon word* if it satisfies one of the following two equivalent conditions:

L1) w is smaller than any of its cyclic rotations.

L2) w is either a single letter or $w = uv$ for Lyndon words u, v with $u < v$.

Let $I_L^* \subset I^*$ be the set of Lyndon words. To any Lyndon word w, we can associate an element $\iota(w)$ of $F(I)$ as follows:

ι) If w is a letter in I, put $\iota(w) = w$.

ι') If $w = uv$ for non-empty Lyndon words u, v with $u < v$ and v of maximal possible length, put $\iota(w) := [\iota(u), \iota(v)]$.

We claim that $\{\iota(w)\}_{w \in I_L^*}$ is a basis for $F(I)$.

(a) Show that $\iota(w) \in w + \sum\limits_{w' > w} \mathbb{Z}w'$ for every Lyndon word w.

(b) Show that $\{\iota(w_1) \cdots \iota(w_l) \mid l \geq 0,\ w_1, ..., w_l \in I_L^*,\ w_1 \geq w_2 \geq ... \geq w_l\}$ is a basis for $T(I)$; you may use here that any word w in I can be written in a unique way as the product $w_1 \cdots w_l$ with Lyndon words $w_1, ..., w_l$ satisfying $w_1 \geq ... \geq w_l$.

(c) Use the PBW-Theorem and $\mathcal{U}(F(I)) \cong T(I)$ to conclude that $\{\iota(w)\}_{w \in I_L^*}$ is a basis for $F(I)$.

(+4 points)