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Sheet 6

Hand in on Thursday, 22th November, prior to the lecture.

Exercise 1

Let $R \subseteq V$ be a root system. Show how to construct the Dynkin diagram of R from the Cartan matrix and vice versa. (4 points)

Exercise 2

Compute the order of the Weyl groups of the root systems of type A_n, B_n, C_n, D_n (see the last exercise on Sheet 5 for concrete realizations of these). (4 points)

Exercise 3

Let $R \subseteq V$ be a root system with some basis Δ. Show that $\Delta^\vee := \{\alpha^\vee \mid \alpha \in \Delta\}$ is a basis for the dual root system. (4 points)

Exercise 4

Let V be a finite dimensional real vector space. A *lattice* is a discrete subgroup $Q \subset V$ which spans V over $\mathbb{R}$. Equivalently (why?), a lattice is a subgroup $Q \subset V$ of the form

$$\left\{ \sum \lambda_i \beta_i \;\middle|\; \lambda_i \in \mathbb{Z} \right\}$$

where $\{\beta_i\}_{i=1}^n$ is a basis of V.
Assume that V is equipped with a positive definite inner product $(-,-)$. A lattice $Q \subset V$ is called *integral* if $(\alpha, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in Q$. A lattice $Q \subset V$ is called *even* if $(\alpha, \alpha) \in 2\mathbb{Z}$ for all $\alpha \in Q$.

i) Prove that an even lattice is integral.

ii) Let $Q \subset V$ be an even lattice. Assume that the set $R_Q = \{\alpha \in Q \mid (\alpha, \alpha) = 2\}$ spans V. Show that R_Q is a root system in V.

iii) Let $V = \bigoplus_{i=1}^r \mathbb{R} e_i$ equipped with the standard inner product $(e_i, e_j) = \delta_{ij}$. Consider

$$\Gamma_r = \left\{ \sum a_i e_i \;\middle|\; \begin{array}{c} \text{either all } a_i \in \mathbb{Z} \\ \text{or all } a_i \in \mathbb{Z} + \frac{1}{2} \end{array} \text{ and } \sum a_i \in 2\mathbb{Z} \right\}.$$

Prove that Γ_r is an even lattice if and only if r is divisible by 8.

iv) Consider $\Gamma = \Gamma_8 \subset V = \mathbb{R}^8$. Show that V is spanned by vectors $v \in \Gamma$ such that $(v, v) = 2$. Describe the roots in R_Γ.

v) Find a basis for R_Γ and conclude that R_Γ is a root system of type E_8. (*Hint:* Consider the functional $t \in V^*$ given by

$$t = \sum_{i=1}^{7} (i-1)e_i^* + 23e_8^*.$$

where e_i^* is the dual basis of $\{e_i\}_{i=1}^n$ (i.e. $\langle e_i^*, e_j\rangle = \delta_{ij}$ for all i, j), and look at the set of roots $\alpha \in R_\Gamma$ with $\langle t, \alpha\rangle = 1$.)

v) Let H_7 be the hyperplane orthogonal to $e_7 + e_8$. Show that $R \cap H_7$ is a root system of type E_7.

vi) Let $H_6 \subset R$ be the subspace orthogonal to $e_6 + e_7 + 2e_8$ and $e_7 + e_8$. Show that $R \cap H_6$ is a root system of type E_6.

We have now constructed all the irreducible root systems! (4 points)