Exercises for Algebra II
Bonn, WS 2012/13

Dr. Geordie Williamson
Hanno Becker

Sheet 5

Hand in on Thursday, 15th November, prior to the lecture.

Exercise 1

Let V be a finite dimensional real vector space and $R \subset V$ a root system. Fix two roots $\alpha, \beta \in R$. In this exercise we analyse the possible angles between α and β. Recall from the lectures that we may equip V with a positive definite bilinear form $(-,-)$ such that each reflection s_α is orthogonal with respect to this form. We also saw that we have

$$\langle \alpha^\vee, \beta \rangle = 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}. \qquad (*)$$

a) Let ϕ denote the angle between α and β. Show the relation

$$\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle = 4 \cos^2 \phi$$

Conclude that $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle \in \{0, 1, 2, 3, 4\}$. (*Hint:* Use $(*)$ together with the fact that, in an Euclidean space, we have $(v, w) = \|v\| \cdot \|w\| \cdot \cos \phi$, where ϕ denotes the angle between v and w.) (1 point)

b) Show that, if α and β are not proportional, we have

$$\phi \in \{\pi/2, \pi/3, 2\pi/3, \pi/4, 3\pi/4, \pi/6, 5\pi/6\}.$$

What can be said about the ratios of the lengths of α and β in each case? (1 point)

c) Describe $(\mathbb{R}\alpha + \mathbb{R}\beta) \cap R$ and deduce that any rank 2 root system is isomorphic to $A_1 \times A_1$, A_2, B_2 or G_2 (as claimed in lectures). (1 point)

d) Show that $\langle \alpha^\vee, \beta \rangle > 0$ and α and β are not proportional then $\beta - \alpha$ is a root. (1 point)

Exercise 2

Let $R \subset V$ be a root system. Show that the set $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$ is a root system in V^*. (We call $R^\vee$ the *dual root system* to R). (4 points)

Exercise 3

Let $R \subset V$ be a reduced root system and $\Delta := \{\alpha_1, ..., \alpha_n\} \subset R$ be a basis of R with associated Cartan matrix $A := (\alpha_i^\vee(\alpha_j))_{1 \leq i,j \leq n}$.

(a) Show that for each positive root β (relative to Δ) there exists a sequence $i_1, ..., i_k$ in $\{1, 2, ..., n\}$ such that each partial sum $\alpha_{i_1} + ... + \alpha_{i_j}$ (for $1 \leq j \leq k$) is a root, and $\beta = \alpha_{i_1} + ... + \alpha_{i_k}$. (2 points)

(b) Describe an algorithm to reconstruct the set of roots from Δ and A. (2 points)

Exercise 4

Given any of the pairs (V, R) in the list below, prove that R is a root system in V, construct an explicit basis and compute the corresponding Cartan matrix and Dynkin diagram. (4 points)

A_n: $V := \{\sum_{i=1}^n a_i e_i \in \mathbb{R}^n \mid \sum_{i=1}^n a_i = 0\}$, $R := \{e_i - e_j \mid 1 \leq i \neq j \leq n\}$.

D_n: $V := \mathbb{R}^n$, $R := \{\pm e_i \pm e_j \mid 1 \leq i \neq j \leq n\}$.

B_n: $V := \mathbb{R}^n$, $R := \{\pm e_i, \pm e_i \pm e_j \mid 1 \leq i \neq j \leq n\}$.

C_n: $V := \mathbb{R}^n$, $R := \{\pm 2e_i, \pm e_i \pm e_j \mid 1 \leq i \neq j \leq n\}$.