Exercises for Algebra II — Dr. Geordie Williamson
Bonn, WS 2012/13 — Hanno Becker

Sheet 4

Hand in on Thursday, 8th November, prior to the lecture.

This exercise sheet is about the classification of finite-dimensional simple modules over the Lie-algebra $\mathfrak{sl}_2(k)$ for a field k of characteristic 0. It is long and sometimes tricky, but still *fun and very important*, so take your time and work on it!

Recall: If we let $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ then e, f and h give a basis of $\mathfrak{sl}_2(k)$ with relations $[h,e] = 2e$, $[h,f] = -2f$ and $[e,f] = h$. Hence, a representation of $\mathfrak{sl}_2(k)$ consists of a vector space V over k together with three endomorphisms E, F and H satisfying $HE - EH = 2E$, $HF - FH = -2F$ and $EF - FE = H$. (We recover the representation $\phi : \mathfrak{sl}_2(k) \to \mathfrak{gl}(V)$ by setting $\phi(e) = E$, $\phi(f) = F$ and $\phi(h) = H$.) In this exercise we always assume that V is *finite dimensional.*

Exercise 1

The crucial idea in understanding $\mathfrak{sl}_2(k)$-modules is to look at the eigenspace decomposition of H and to check how E and F affect it. The basis for this are some commutator relations:

a) Show that the endomorphisms E and H satisfy the relation

$$(H - (\lambda + 2))^n E = E(H - \lambda)^n.$$

(Here $\lambda \in k$ and we write λ instead of $\lambda \cdot \mathrm{id}_V$.) Deduce that if $v \in V$ belongs to the generalised λ-eigenspace of H, then Ev belongs to the generalised $(\lambda+2)$-eigenspace.

(2 points)

b) Deduce a similar statement for the action of F on the generalised eigenspaces of H.

(1 point)

c) Show that the direct sum $W(V)$ of all generalized H-eigenspaces is an $\mathfrak{sl}_2(k)$-submodule of V. (2 points)

Exercise 2

Next we check that V decomposes into generalized H-eigenspaces, i.e. $V = W(V)$, a fact which is of course for free if $k = \overline{k}$. The actual proof is containd in Exercise 3, while here we provide some useful commutator relations, which will also be needed in Exercise 4.

a) Show the relation (for $n \geq 1$) $HF^n = F^nH - 2nF^n$. (1 point)

b) Show ($n \geq 1$ as before) $EF^n = F^nE + nF^{n-1}H - n(n-1)F^{n-1}$. (1 point)

c) Deduce that, if $v \in V$ is a vector such that $Ev = 0$ then

$$E^nF^nv = nE^{n-1}F^{n-1}(H-(n-1))v = n!\prod_{i=1}^{n}(H-(i-1))v.$$

(1 point)

d) Use (c) to show that H acts diagonalizably on $W(V)$ (*Hint:* Start with generalized eigenvectors $v \in V_\lambda$ satisfying $Ev = 0$.) (1 point)

Exercise 3

Here we finish the proof that $W(V) = V$.

a) Show that E, F act nilpotently on V. (*Hint:* Use Lie's theorem over the algebraic closure of k, applied to suitable solvable subalgebras of $\mathfrak{sl}_2(k)$. If you like a more direct argument, use 2(a) to check that all powers E^n and F^n have trace 0 on V, and deduce the nilpotency of E, F) (2 points)

b) Check that $\mathrm{kern}(E) \neq 0$ and that H preserves it. Then use (a), 2(c) to show that H acts diagonalizably on $\mathrm{kern}(E)$, with non-negative, integral eigenvalues. In particular, there is at least one eigenvalue for H, as claimed, so $W(V) \neq 0$. (1 point)

c) Check that $W(V/W(V)) = 0$, and deduce $V = W(V)$. (1 point)

Exercise 4

a) Let $v \in V_\lambda$ be an H-eigenvector such that $Ev = 0$. We know by Exercise 3(b) (or rather 2(c)) that $\lambda \in \mathbb{Z}_{\geq 0}$. Show the relations:

$$HF^nv = (\lambda - 2n)F^nv,$$
$$EF^nv = n(\lambda-(n-1))F^{n-1}v.$$

Deduce that $F^{\lambda+1}v = 0$ and that the F^iv for $0 \leq i \leq \lambda$ are linearly independent and span a simple submodule of V. (2 points)

b) Check that the above relations define an $\mathfrak{sl}_2(k)$-module for any non-negative integer l. Deduce that there is (up to isomorphism) a unique simple module $V(l)$ of dimension $l+1$ for all non-negative integers l. (2 points)