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Sheet 2

Hand in on Monday, 22th October, prior to the lecture.

Exercise 1

Let k be a field and $\mathfrak{sl}_2(k) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{gl}_2(k) \;\middle|\; a + d = 0 \right\}$.

(a) Find the structure constants of $\mathfrak{sl}_2(k)$ with respect to the basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(1 point)

(b) Determine the ideals of $\mathfrak{sl}_2(k)$. Is $\mathfrak{sl}_2(k)$ simple? (3 points)

Exercise 2

Let $\mathfrak{g}$ be a Lie algebra. In the lectures it was explained how to pass between a $\mathfrak{g}$-module V and a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. Prove that these two procedures are inverse to one another. (4 Points)

Exercise 3

Let k be a field.

(a) Let $\mathfrak{g}$ be a 3-dimensional Lie algebra over k with $\dim_k(\mathfrak{g}') = 1$ and $\mathfrak{g}' \subset \mathfrak{z}(\mathfrak{g})$. Determine the structure constants of $\mathfrak{g}$ with respect to a suitable basis, and show that there is up to isomorphism a unique such algebra. (This Lie algebra is the famous *Heisenberg algebra*.) Find an isomorphism of $\mathfrak{g}$ with a Lie subalgebra of $\mathfrak{gl}_3(k)$. (2 points)

(b) Classify all 3-dimensional Lie-algebras $\mathfrak{g}$ over k with $\mathfrak{z}(\mathfrak{g}) \neq 0$. (2 points)

Exercise 4

Prove that $\mathfrak{sl}_n(\mathbb{C})$ is simple. (4 points)