Refined Donaldson–Thomas theory

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Reading group on
Stability conditions, DT invariants and their geometry
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Goal of the talk

Give a basic introduction to the refined Donaldson–Thomas theory of Kontsevich and Soibelman.

This talk is largely independent of the previous three talks.

It is in some sense a continuation of Séverin’s talk, although

- I will take a cohomological, as opposed to motivic, approach, and
- I will focus on quiver theoretic, as opposed to sheaf theoretic, examples.
Motivation

- Let $M$ be a compact oriented 3-manifold and $G$ a compact Lie group. The Chern–Simons functional is

\[ CS(A) = \int_M \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \quad A \in \Omega^1(M; \mathfrak{g}). \]

Critical points of $CS$ are flat $G$-connections, i.e. $(d + A)^2 = 0$. Chern–Simons theory is about “counting” flat $G$-bundles.

- Let $X$ be a compact Calabi–Yau threefold. The holomorphic Chern–Simons functional is

\[ CS^{\text{hol}}(\mathcal{A}) = \int_X \text{tr}(\mathcal{A} \wedge \bar{\partial}A + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}) \wedge \Omega^{3,0}, \quad \mathcal{A} \in \Omega^{0,1}(X; \mathfrak{g}_\mathbb{C}). \]

Critical points of $CS^{\text{hol}}$ are holomorphic $G_\mathbb{C}$-bundles, i.e. $(\bar{\partial} + \mathcal{A})^2 = 0$. Donaldson–Thomas theory should be about “counting” holomorphic $G_\mathbb{C}$-bundles.
Thomas developed an algebraic, rather than analytic, approach to the above idea in the case $\mathbb{G}_C = \text{GL}_d(C)$, i.e. vector bundles.

- Let $\nu \in H^{\text{even}}(X)$. Denote by $\mathcal{M}^\text{st}_\nu$ and $\mathcal{M}^\text{ss}_\nu$ the moduli spaces of (semi)stable sheaves on $X$ with Chern character $d$.

- Under the assumption $\mathcal{M}^\text{st}_\nu = \mathcal{M}^\text{ss}_\nu$, Thomas constructed a virtual fundamental class $[\mathcal{M}^\text{ss}_\nu]^\text{vir} \in A_0(\mathcal{M}^\text{ss}_\nu)$ and defined the numerical Donaldson–Thomas invariant to be

$$\Omega_{X,\nu} = \int_{[\mathcal{M}^\text{ss}_\nu]^\text{vir}} 1 \in \mathbb{Z}.$$ 

- This can also be written in terms of the Behrend function

$$\Omega_{X,\nu} = \chi(\mathcal{M}^\text{ss}_\nu, \nu_{\mathcal{M}^\text{ss}_\nu}).$$

- This approach fails if there are strictly semistable sheaves.

Classical analogue: Atiyah–Bott’s computation via Yang–Mills theory of the cohomology of the moduli space of stable vector bundles over a curve with coprime rank and degree.
Why refine?

- Unless \( \nu \) is chosen in a very special way, there usually exist strictly semistable objects. For example, if \( \nu \) is good, then \( 2\nu \) is not. \textit{A priori}, this deficiency seems unrelated to refinement.

- Aside from providing more refined information, e.g. computing the Poincaré polynomial of \( \mathcal{M}_\nu^{ss} \) instead of its Euler characteristic, more structure is expected in the refined theory.

- Physically, refinement corresponds to keeping track of the spin of the BPS particles.

Joyce–Song and Kontsevich–Soibelman developed independent approaches to address both issues. We will talk about the latter approach today.
What additional structure is expected in the refined theory?

- Harvey–Moore suggested that single particle BPS states in a theory with extended supersymmetry (e.g. $\mathcal{N} = 2$) form a Lie algebra via scattering.

- Physical constructions of the BPS Lie algebra were given in some special cases, but none were entirely satisfactory.

- The ideas of Harvey–Moore were motivation for Kontsevich–Soibelman to introduce the cohomological Hall algebra. However, this is really an algebra of multiparticle BPS states, and so does not itself realize the BPS Lie algebra.
Background and reminders
Let $X$ be a variety (over $\mathbb{C}$) with an action of a linear algebraic group $G$. There is an associated quotient stack $[X/G]$ whose (rational) cohomology is the $G$-equivariant cohomology of $X$:

$$H^\bullet([X/G]) \cong H^\bullet_G(X; \mathbb{Q}).$$

**Example (classifying stack)**

When $X = \text{pt}$, write $BG$ for $[\text{pt}/G]$. Then we have

$$H^\bullet(BG) \cong H^\bullet_G(\text{pt}),$$

the $G$-equivariant cohomology of a point.

More generally, if $X$ is $G$-equivariantly contractible, then

$$H^\bullet([X/G]) \cong H^\bullet_G(\text{pt}).$$
Only examples we will need today:

1. \( G = \mathbb{C}^\times: \)

\[
H^\bullet(B\mathbb{C}^\times) \simeq H^\bullet(\mathbb{C}P^\infty) \simeq \mathbb{Q}[x], \quad |x| = 2.
\]

Explicitly,
\[
x = c_1(\text{pt} \times_{\mathbb{C}^\times} \mathbb{C}_1 \to B\mathbb{C}^\times).
\]

2. \( G = \mathbb{T}^n = (\mathbb{C}^\times)^n: \)

\[
H^\bullet(B\mathbb{T}^n) \simeq \mathbb{Q}[x_1, \ldots, x_n], \quad |x_i| = 2.
\]
3. $G = \text{GL}_n$: Pullback along the inclusion $\triangleq \mathbb{T}^n \hookrightarrow \text{GL}_n$ is a homomorphism $H^\bullet(B\text{GL}_n) \to H^\bullet(B\mathbb{T}^n)$ which induces isomorphisms

$$H^\bullet(B\text{GL}_n) \cong H^\bullet(B\mathbb{T}^n)^{S_n} \cong \mathbb{Q}[x_1, \ldots, x_n]^{S_n}.$$ 

4. Let $\text{GL}_{n,m} \leq \text{GL}_{n+m}$ be the subgroup which preserves $\mathbb{C}^n \subset \mathbb{C}^{n+m}$:

$$\text{GL}_{n,m} = \begin{pmatrix} \text{GL}_n & \text{Hom}_\mathbb{C}(\mathbb{C}^m, \mathbb{C}^n) \\ 0 & \text{GL}_m \end{pmatrix}.$$ 

The morphism $\text{GL}_{n,m} \to \text{GL}_n \times \text{GL}_m$ induces an isomorphism

$$H^\bullet(B(\text{GL}_n \times \text{GL}_n)) \xrightarrow{\sim} H^\bullet(B\text{GL}_{n,m})$$
A quiver $Q$ consists of nodes $Q_0$ and arrows $Q_1$.

- $\text{Rep}_\mathbb{C}(Q)$, the abelian category of representations of $Q$
- $\Lambda^+_Q = \mathbb{Z}_{\geq 0}Q_0$, the monoid of dimension vectors of $Q$
- The dimensions of $V \in \text{Rep}_\mathbb{C}(Q)$ are
  \[ \text{dim } V \in \Lambda^+_Q, \quad \text{dim } V \in \mathbb{Z}_{\geq 0} \]

- The Euler form of $Q$ is
  \[ \chi(U, V) = \dim \text{Hom}_{\text{Rep}(Q)}(U, V) - \dim \text{Ext}^1_{\text{Rep}(Q)}(U, V) \]
  \[ = \sum_{i \in Q_0} d_i e_i - \sum_{(i \xrightarrow{\alpha} j) \in Q_1} d_i e_j \]

- $Q$ is symmetric if $\chi : \Lambda^+_Q \times \Lambda^+_Q \to \mathbb{Z}$ is symmetric (if and only if $Q$ is symmetric in the combinatorial sense)
The variety of representations of $Q$ of dimension vector $d \in \Lambda^+_Q$ is

$$R_d = \prod_{(i \rightarrow j) \in Q_1} \text{Hom}_\mathbb{C}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}).$$

It has a linear action of the group

$$\text{GL}_d = \prod_{i \in Q_0} \text{GL}_{d_i}$$

by change of basis whose orbits are isomorphism classes of representations. The diagonal $\mathbb{C}^\times \leq \text{GL}_d$ acts trivially on $R_d$. Note that

$$\dim R_d - \dim \text{GL}_d = -\chi(d, d).$$
Fix a stability \( \theta \in \text{Hom}_\mathbb{Z}(\mathbb{Z}Q_0, \mathbb{Z}) \), i.e. an integer for each node \( Q_0 \). The slope of \( 0 \neq V \in \text{Rep}(Q) \) is

\[
\mu(V) = \frac{\theta(\dim V)}{\dim V} \in \mathbb{Q}.
\]

**Definition (Stability)**

A non-zero representation \( V \) is stable (semistable) if

\[
\mu(U) < \mu(V), \quad \text{(resp. } \mu(U) \leq \mu(V)\text{)}
\]

for any subrepresentation \( 0 \nsubseteq U \nsubseteq V \).

(Semi)stability defines \( \text{GL}_d \)-stable open subvarieties

\[
R_d^{\theta-\text{st}} \subset R_d^{\theta-\text{ss}} \subset R_d.
\]

Key point: A stable representation has automorphism group \( \mathbb{C}^\times \), arising as the diagonal in \( \text{GL}_d \).
(Harder–Narasimhan filtrations) Any $V \in \text{Rep}(Q)$ admits a unique filtration $0 \subset V_1 \subset \cdots \subset V_n = V$ such that each $V_i/V_{i-1}$ is semistable and

$$
\mu(V_1/V_0) > \cdots > \mu(V_n/V_{n-1}).
$$

(Jordan–Hölder filtrations) Any semistable $V \in \text{Rep}(Q)$ admits a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ such that each $V_i/V_{i-1}$ is stable and

$$
\mu(V_1/V_0) = \cdots = \mu(V_n/V_{n-1}).
$$

Moreover, the set $\{V_1/V_0, \ldots, V_n/V_{n-1}\}$ of Jordan–Hölder factors is well-defined up to isomorphism.
Example

For most of today, we can take $\theta = 0$. In this case:

- semistability is a vacuous condition, so that $R_d^{ss} = R_d$
- stability is simplicity/irreducibility, i.e. the representation has no non-trivial subrepresentations
- The Harder–Narasimhan filtration of a representation $V$ is simply itself:
  \[ 0 = V_0 \subset V_1 = V \]

Let $U, V$ be simple representations. Then

\[ 0 \leftrightarrow U \leftrightarrow U \oplus V \]

and

\[ 0 \leftrightarrow V \leftrightarrow U \oplus V \]

are distinct Jordan–Hölder filtrations of $U \oplus V$. The Jordan–Hölder factors are $U$ and $V$. 
Moduli stacks of (semi)stable representations:

\[ \mathcal{M}^\theta_{d-ss} = \left[ R^\theta_{d-ss} / \text{GL}_d \right] \]

and

\[ \mathcal{M}^\theta_{d-st} = \left[ R^\theta_{d-st} / \text{GL}_d \right]. \]

Geometric invariant theory gives rise to moduli schemes of (semi)stable representations:

\[ \mathcal{M}^\theta_{d-ss} = R_d / \theta \text{GL}_d \]

and

\[ \mathcal{M}^\theta_{d-st} = R^\theta_{d-st} / \text{GL}_d = \{ \text{stable reps} \} / \text{iso}. \]

- \( \mathcal{M}^\theta_{d-st} \subset \mathcal{M}^\theta_{d-ss} \) is open and, if non-empty, dense
- \( \mathcal{M}^\theta_{d-st} \) is smooth, but usually not projective
- \( \mathcal{M}^\theta_{d-ss} \) is a partial compactification of \( \mathcal{M}^\theta_{d-st} \) which, however, is usually singular
The description of $\mathcal{M}^{\theta-ss}_d$ is complicated by the fact that $GL_d$-orbits on $R^{\theta-ss}_d$ need not be closed.

- $\mathcal{M}^{\theta-ss}_d$ parameterizes closed $GL_d$-orbits in $R^{\theta-ss}_d$
- In particular, $\mathcal{M}^{\theta-ss}_d$ does not parameterize isomorphism classes of semistable representations
- Instead, $\mathcal{M}^{\theta-ss}_d$ parameterizes isomorphism classes of polystable representations

**Definition (Polystability)**

A representation is polystable if it is the direct sum of stable representations of the same slope.

Hence,

$$\mathcal{M}^{\theta-ss}_d = R_d/\theta GL_d = R^{\theta-ss}_d / S\text{-equivalence} = \{\text{polystable reps}\}/\text{iso}$$

Explicitly, a semistable representation $V$ is sent to the sum of its Jordan–Hölder factors

$$\bigoplus_{i=1}^n V_i/V_{i-1}.$$
Morphisms to coarse moduli schemes:

- The map
  \[ \mathcal{M}_d^{\theta-ss} \rightarrow \mathcal{M}_d^{\theta-ss}, \quad V \mapsto JH(V) \]
  is very difficult to study. Roughly, \( \mathcal{M}_d^{\theta-ss} \) is easy to understand, but it is non-geometric, while \( \mathcal{M}_d^{\theta-ss} \) is difficult to understand.

- The map
  \[ \mathcal{M}_d^{\theta-st} \rightarrow \mathcal{M}_d^{\theta-st} \]
  is a \( \mathbb{C}^\times \)-gerbe. In particular,
  \[ H^\bullet(\mathcal{M}_d^{\theta-st}) \xrightarrow{\sim} H^\bullet(\mathcal{M}_d^{\theta-st}) \otimes_{\mathbb{Q}} H^\bullet(B\mathbb{C}^\times) \cong H^\bullet(\mathcal{M}_d^{\theta-st}) \otimes_{\mathbb{Q}} \mathbb{Q}[z]. \]
  Roughly, understanding \( \mathcal{M}_d^{\theta-st} \) and \( \mathcal{M}_d^{\theta-st} \) is of the same level of difficulty (which is high).
Donaldson–Thomas theory
A heuristic path to refined DT invariants

Consider the case $\theta = 0$ (or fix $\theta$ and a slope $\mu \in \mathbb{Q}$ and consider only dimension vectors of slope $\mu$).

There is a map

$$\mathcal{M}_{d_1}^{st} \times \cdots \times \mathcal{M}_{d_n}^{st} \overset{\oplus}{\to} \mathcal{M}_{d_1+\cdots+d_n}^{ss}.$$ 

This is invariant under permutations and so, setting

$$\mathcal{M}^{st} = \bigsqcup_{d \in \Lambda^+_Q} \mathcal{M}^{st}_d, \quad \mathcal{M}^{ss} = \bigsqcup_{d \in \Lambda^+_Q} \mathcal{M}^{ss}_d,$$

we obtain a map

$$\text{Sym } \mathcal{M}^{st} \overset{\oplus}{\to} \mathcal{M}^{ss}.$$ 

This is a bijection of the underlying sets of complex points, but it is not a map of schemes! Basic problem: Jordan–Hölder filtrations are not unique. For example,

$$\text{Sym } H^\bullet(\mathcal{M}^{st}) \ncong H^\bullet(\mathcal{M}^{ss}).$$
Key idea of Donaldson–Thomas theory

After passing to stacks, this isomorphism should hold.

At the level of stacks, there is a morphism

\[ \text{Sym } \mathcal{M}^\text{st} \xrightarrow{\oplus} \mathcal{M}^\text{ss}. \]

This is no longer a bijection in any sense; the right hand side sees extensions, the left does not. If it were an isomorphism, we would expect

\[ \text{Sym } H^\bullet(\mathcal{M}^\text{st}) \cong H^\bullet(\mathcal{M}^\text{ss}), \]

or

\[ \text{Sym } (H^\bullet(\mathcal{M}^\text{st}) \otimes \mathbb{Q}[z]) \cong H^\bullet(\mathcal{M}^\text{ss}). \]
To turn these heuristics into a definition, define

\[ \mathcal{H}^\theta_{Q} = \bigoplus_{d \in \Lambda^+} H^\bullet(\mathcal{M}_d^\theta). \]

Put a \( \Lambda^+ \times \mathbb{Z} \)-grading on \( \mathcal{H}_Q \) by setting

\[ \mathcal{H}^\theta_{Q,(d,k)} = H^{k+\dim \mathcal{M}_d^\theta}(\mathcal{M}_d^\theta) = H_{\text{GL}_d}^{k-\chi(d,d)}(R_d^\theta). \]

For example, if \( \theta = 0 \), then

\[ \mathcal{H}_{Q,(d,k)} = \left\{ \text{degree } k - \chi(d,d) \text{ symmetric polynomials in } x_{i,1}, \ldots, x_{i,d}, i \in Q_0 \right\}. \]
For the remainder of the talk, we will assume that $Q$ is symmetric.

**Definition**

1. The refined Donaldson–Thomas invariant of $Q$ is a $\Lambda_Q^+ \times \mathbb{Z}$-graded vector space $V_{Q,\theta}^{\text{prim}}$ which satisfies

   $$\mathcal{H}_{Q,\theta}^{\text{ss}} \simeq \text{Sym}^{gr} \left( V_{Q,\theta}^{\text{prim}} \otimes \mathbb{Q}[z] \right),$$

   where $\deg(z) = (0, 2) \in \Lambda_Q \times \mathbb{Z}$.

2. The refined Donaldson–Thomas series of $Q$ is

   $$\Omega_Q^{\theta} \left( q^{\frac{1}{2}}, t \right) = \sum_{(d,k) \in \Lambda_Q^+ \times \mathbb{Z}} \dim V_{Q,(d,k)}^{\text{prim},\theta} (-q^{\frac{1}{2}})^k t^d.$$

Our heuristic:

$$H^\bullet(\mathcal{M}^{ss}) \simeq \text{Sym} \left( H^\bullet(\mathcal{M}^{st}) \otimes \mathbb{Q}[z] \right).$$
Define

\[ A_\theta^Q(q^{\frac{1}{2}}, t) = \sum_{(d,k) \in \Lambda_Q^+ \times \mathbb{Z}} \dim \mathcal{H}_{Q,(d,k)}^{\theta-ss} (-q^{\frac{1}{2}})^k t^d. \]

**Proposition (original definition of Kontsevich–Soibelman)**

The following equality holds:

\[ A_\theta^Q(q^{\frac{1}{2}}, t) = \prod_{(d,k) \in \Lambda_Q^+ \times \mathbb{Z}} (q^{\frac{k}{2}} t^d ; q)^{-\Omega^\theta_{Q,(d,k)}}. \]

This equality can be used to compute \( \Omega^\theta_Q \) from \( A_\theta^Q \).
Note that if $\theta = 0$, then we have

$$A_Q(q^{\frac{1}{2}}, t) = \sum_{(d,k) \in \Lambda_Q^+ \times \mathbb{Z}} \dim \mathcal{H}_{Q,(d,k)}(-q^{\frac{1}{2}})^k t^d$$

because $\theta = 0$)

$$= \sum_{d \in \Lambda_Q^+} \frac{(-q^{\frac{1}{2}}) \chi(d,d) \prod_{i \in Q_0} \prod_{j=1}^{d_i} (1 - q^j)}{\prod_{i \in Q_0} \prod_{j=1}^{d_i} (1 - q^j)} t^d.$$

For general $\theta$, we can compute $A_Q^{\theta-ss}$ from $A_Q$ using the Harder–Narasimhan filtration. In this way, $\Omega_Q^\theta$ is indeed computable.
For simplicity, I’ll often omit $\theta$ from the notation.

Proposition: $A_Q(q^{\frac{1}{2}}, t) = \prod_{(d,k) \in \Lambda_Q^+ \times \mathbb{Z}} (q^{\frac{k}{2}} t^d; q)^{-\Omega_{Q,(d,k)}}$.

Proof.

The $q$-Pochhammer symbol is

$$(t; q)_\infty = \prod_{i \geq 0} (1 - q^i t).$$

Let $x \in V_{Q,(d,k)}^{\text{prim}}$ be of degree $(d, k)$. The symmetric algebra on $x \cdot z^i$ has Poincaré series

$$\begin{cases} 1 - q^{\frac{k}{2}+i} t^d & \text{if } k \text{ is odd} \\ (1 - q^{\frac{k}{2}+i} t^d)^{-1} & \text{if } k \text{ is even} \end{cases}$$

Hence $x \otimes \mathbb{Q}[z]$ contributes a factor of $(q^{\frac{k}{2}} t^d; q)_\infty^{-\text{sgn}(k)}$. □
We introduced $V^\text{prim}_Q$ as a “DT-corrected” version of $H^\bullet(\mathcal{M}^\text{st})$. For this to be reasonable, the following should be true.

**Conjecture**

1. (Integrality) For each $d \in \Lambda^+_Q$, the summand $V^\text{prim}_{Q,d}$ is finite dimensional.

2. (Geometricity) The vector space $V^\text{prim}_Q$ has geometric meaning.

If the Integrality Conjecture is true, then we can define numerical Donaldson–Thomas invariants (i.e. a “DT-corrected” Euler characteristic of $\mathcal{M}^\text{st}$) by

$$
\Omega^\text{num}_Q(t) = \Omega_Q(q^{\frac{1}{2}} = 1, t).
$$

A good proof of Geometricity should imply Integrality.
A representation is a vector space.

Two representations of the same dimension are isomorphic.

Any representation of dimension one is stable, and these are the only stable representations (vector spaces of dimension greater than one have a non-trivial subspace).

It follows that

\[ M_{ss}^1 = \text{pt}, \quad M_{ss}^2 = \text{pt}, \quad \ldots \quad M_{ss}^d = \text{pt}, \]

and

\[ M_{st}^1 = \text{pt}, \quad M_{st}^2 = \emptyset, \quad \ldots \quad M_{st}^d = \emptyset. \]

A direct computation shows

\[ A_{L_0}(q^{\frac{1}{2}}, t) = (q^{\frac{1}{2}} t; q)_{\infty}, \]

giving

\[ \Omega_{L_0} = -q^{\frac{1}{2}} t. \]
A representation is a vector space with an endomorphism.

Any representation of dimension one is stable, and these are the only stable representations (any \( A \in M_{d \times d}(\mathbb{C}) \) has a non-zero eigenvector).

It follows that

\[ M_1^{st} = \mathbb{C}, \quad M_2^{st} = \emptyset, \quad \ldots \quad M_d^{st} = \emptyset. \]
Example: $Q = L_1$ (ctd)

Let’s describe $\mathcal{M}_d^{ss}$. Let $A \in R_d = M_{d \times d}$.

- The Jordan canonical form of $A$ is
  
  $$A \sim_{GL_d} J_{d_1}(\lambda_1) \oplus \cdots \oplus J_{d_n}(\lambda_n).$$

- Note that
  
  $$\text{diag}(\xi^{d_i^{-1}}, \ldots, \xi, 1) \cdot J_{d_i}(\lambda_i) \cdot \text{diag}(\xi^{d_i^{-1}}, \ldots, \xi, 1)^{-1} = J_{d_i}^\xi(\lambda_i).$$

- Taking $\xi \to 0$ shows that the $GL_d$-orbit closure of $A$ contains $D_{d_1}(\lambda_1) \oplus \cdots \oplus D_{d_n}(\lambda_n)$.

- The inclusion $D_{d \times d} \hookrightarrow M_{d \times d}$ induces an isomorphism
  
  $$\mathbb{C}^d \simeq D_{d \times d}//\mathfrak{S}_d \sim M_{d \times d}//GL_d = \mathcal{M}_d^{ss}.$$

It follows that

$$\mathcal{M}_1^{ss} = \mathbb{C}, \quad \mathcal{M}_2^{ss} = \mathbb{C}^2, \quad \ldots \quad \mathcal{M}_d^{ss} = \mathbb{C}^d.$$

In particular, all the moduli spaces are smooth—this is an accident!
Summarizing, we have

\[ M_1^{ss} = \mathbb{C}, \quad M_2^{ss} = \mathbb{C}^2, \quad \ldots \quad M_d^{ss} = \mathbb{C}^d \]

and

\[ M_1^{st} = \mathbb{C}, \quad M_2^{st} = \emptyset, \quad \ldots \quad M_d^{st} = \emptyset. \]

A direct computation shows

\[ A_{L_1}(q^{\frac{1}{2}}, t) = (t; q)_\infty^{-1}, \]

giving

\[ \Omega_{L_1} = t. \]
The Integrality Conjecture holds when $\theta = 0$.

There are canonical isomorphisms

$$V_{Q,d}^{\text{prim},\theta} \simeq \begin{cases} IC^\bullet \cdot - \chi(d,d) (M_d^{\theta-\text{st}}) & \text{(Meinhardt–Reineke)} \\ A^\bullet \cdot -\frac{1}{2} \chi(d,d) (M_d^{\theta-\text{st}}) & \text{(Franzen–Reineke)} \\ PH^\bullet \cdot - \chi(d,d) (M_d^{\theta-\text{st}}) & \text{(Chen)} \end{cases}$$

- I don’t know how to prove agreement of the vector spaces on the right hand side without DT theory.
- The first isomorphism seems to be the “right” one, as it generalizes to quivers with potential (Davison–Meinhardt).
- The second result can be used to prove the first.
Cohomological Donaldson–Thomas theory
• Until now, all we have used is the Poincaré series $A_Q$ of $\mathcal{H}_Q$.

• The series $A_Q$ can be obtained in many ways, e.g. working motivically as in Séverin’s lecture:

\[
A^\theta_Q(\mathbb{L}^{\frac{1}{2}}, t) = \int_{\mathcal{H}_Q} [M^{\theta-ss} \xrightarrow{id} M^{\theta-ss}].
\]

• However, the proofs of the above results use additional structure on $\mathcal{H}_Q$, namely, a Hall multiplication.

• The Hall algebra structure on $\mathcal{H}_Q$ is both a useful technical tool and a fundamental object in cohomological Donaldson–Thomas theory.
Recall that
\[ \mathcal{H}_Q = \bigoplus_{d \in \Lambda^+_Q} H^\bullet \chi^{(d,d)}(M_d). \]

Consider the correspondence diagram (as in the finitary and motivic Hall algebras of Séverin’s lecture)

Theorem (Kontsevich–Soibelman)

The map \( p_2! \circ (p_1 \times p_3)^* \) gives \( \mathcal{H}_Q \) the structure of a \( \Lambda^+_Q \times \mathbb{Z} \)-graded associative algebra, called the cohomological Hall algebra (CoHA).
The (ungraded) vector space $\mathcal{H}_Q$ depends only on $Q_0$, but its multiplication depends on $Q$.

**Theorem (Kontsevich–Soibelman)**

Let $Q = L_m$, the $m$-loop quiver and let $f_i \in \mathcal{H}_{Q,d_i}$, $i = 1, 2$. Then

$$f_1 \cdot f_2 = \sum_{\pi \in \mathfrak{S}_{d', d''}} \pi \left( f_1(x'_1, \ldots, x'_{d'}) f_2(x''_1, \ldots, x''_{d''}) \prod_{l=1}^{d''} \prod_{k=1}^{d'} (x''_l - x'_k)^{m-1} \right).$$

- In words, this looks like the standard shuffle product, but with an $m$-dependent integral kernel.
- The proof is a straightforward calculation using Atiyah–Bott localization; the kernel is an equivariant Euler class.
- The formula has an immediate generalization to any quiver.
Theorem (Efimov)

Assume that $\theta = 0$.

Let

$$V_Q^{\text{prim}} \otimes \mathbb{Q}[z] = \mathcal{H}_Q/(\mathcal{H}_Q,+ \cdot \mathcal{H}_Q,+),$$

embedded in $\mathcal{H}_Q$ by the choice of a splitting. Then there is a graded algebra\(^a\) isomorphism

$$\mathcal{H}_Q \simeq \text{Sym}^{\text{gr}}(V_Q^{\text{prim}} \otimes \mathbb{Q}[z]).$$

The Integrality Conjecture holds.

\(^a\)After twisting by a sign; we ignore.

- Both parts are proved using the explicit formula for the CoHA product.
- The most difficult part is to show that $V_Q^{\text{prim}} \otimes \mathbb{Q}[z]$ freely generates $\mathcal{H}_Q$. 
Why should the CoHA product be related to stability?

- Stable representations have no non-trivial subrepresentations.
- Schematically, the Hall algebra multiplication is

\[ [U] \cdot [V] \sim \sum_{W} \langle 0 \to U \to W \to V \to 0 \rangle [W]. \]

- The image of \( \mathcal{H}_{Q,+} \times \mathcal{H}_{Q,+} \to \mathcal{H}_{Q} \) should therefore consist of those representations which have a non-trivial subrepresentation.
- This suggests that the vector space

\[ V_{Q}^{\text{prim}} \otimes \mathbb{Q}[z] \simeq \mathcal{H}_{Q}/(\mathcal{H}_{Q,+} \cdot \mathcal{H}_{Q,+}), \]

that is, the space of minimal generators of \( \mathcal{H}_{Q} \), is related to \( H^{\bullet}(\mathcal{M}^{\text{st}}) \).
The CoHA product reads

$$f_1 \cdot f_2 = \sum_{\pi \in \text{sh}_{d',d''}} \pi \left( f_1(x'_1, \ldots, x'_{d'}) f_2(x''_1, \ldots, x''_{d''}) \prod_{l=1}^{d''} \prod_{k=1}^{d'} (x''_l - x'_k)^{-1} \right).$$

Consider the odd variables $x^i \in \mathcal{H}_{L_0,1}$, $i \geq 0$, of degree $(1, 2i + 1)$. We have

$$x^k \cdot x^l = \frac{x^k_1 x^l_2 - x^l_1 x^k_2}{x_1 - x_2}.$$

If $i = (i_d, \ldots, i_1) \in \mathbb{Z}_{\geq 0}^d$ is a strictly decreasing sequence, then $x^{i_1} \cdots x^{i_d} = s_{i - \delta_d}$, where $s_\lambda$ is the Schur polynomial of a partition $\lambda$ and $\delta_r = (r - 1, \ldots, 1, 0)$. Hence

$$\mathcal{H}_{L_0} \simeq \Lambda[x^0, x^1, x^2, \ldots] = \Lambda[1_1 \cdot \mathbb{Q}[z]]$$

and $V_{L_0}^{\text{prim}} = \mathbb{Q} \cdot 1_1 = \mathbb{Q}(1,1)$. 

Matthew B. Young
Refined DT theory
**Q = L₁ revisited**

The CoHA product reads

\[ f_1 \cdot f_2 = \sum_{\pi \in \mathfrak{h}_{d', d''}} \pi\left(f_1(x'_1, \ldots, x'_{d'}) f_2(x''_1, \ldots, x''_{d''})\right). \]

Consider the even variables \(x^i \in \mathcal{H}_{L_1, 1}, \ i \geq 0\), of degree \((1, 2i)\). We have

\[ x^k \cdot x^l = x_1^k x_2^l + x_1^l x_2^k. \]

In general, \(x^{i_1} \cdots x^{i_d} = N(i) m_i\), where

\[ N(i) = \prod_{k \geq 0} \{j \geq 1 \mid i_j = k\}! \in \mathbb{Z} \]

and \(m_i\) is the monomial symmetric polynomial. Hence

\[ \mathcal{H}_{L_1} \simeq \text{Sym}[x^0, x^1, x^2, \ldots] = \text{Sym}[x^0 \cdot \mathbb{Q}[z]] \]

and \(V^{\text{prim}}_{L_1} = \mathbb{Q} \cdot 1_1 = \mathbb{Q}_{(1,0)}\).
The CoHA product reads

\[ f_1 \cdot f_2 = \sum_{\pi \in \mathcal{S}_{d', d''}} \pi \left( f_1(x'_1, \ldots, x'_{d'}) f_2(x''_1, \ldots, x''_{d''}) \prod_{l=1}^{d''} \prod_{k=1}^{d'} (x''_l - x'_k)^{+1} \right). \]

We have

\[ x^k \cdot x^l = (x^k_1 x^l_2 - x^l_1 x^k_2)(x_1 - x_2) \]

so that \((x_1 + x_2)^n \in \mathcal{H}_{L_2, 2} = \mathbb{Q}[x_1, x_2]^{\mathcal{S}_2}, n \geq 0\), are not in the image of

\[ \mathcal{H}_{L_2, 1} \times \mathcal{H}_{L_2, 1} \rightarrow \mathcal{H}_{L_2, 2}. \]

So

\[ V^\text{prim}_Q = \mathbb{Q} \cdot 1_1 \oplus \mathbb{Q} 1_2 \oplus \cdots \]

and

\[ \Omega_{L_2} = -q^{-\frac{1}{2}} t + q^{-2} t^2 - q^{-\frac{9}{2}} t^3 + q^{-8}(1 + q^2)t^4 + O(t^5). \]

As a check, \(\overline{M}^\text{st}_2 \simeq \mathbb{C}^5\), in agreement with Meinhardt–Reineke.
Idea of the proof of Franzen–Reineke

Recall:

\[ V_{Q,d}^{\text{prim,}\theta} \cong A^\bullet - \frac{1}{2} \chi(d,d)(\mathcal{M}_d^{\theta-\text{st}}). \]

We have seen already many versions of the Hall algebra—finitary, motivic, cohomological. There is also a version for Chow groups:

\[ \mathcal{A}_Q^{\theta-\text{ss}} = \bigoplus_{d \in \Lambda_Q^+} A^\bullet(\mathcal{M}_d^{\theta-\text{ss}})_Q. \]

In fact, the cycle class map \( cl : A^\bullet(\mathcal{M}_d^{\theta-\text{ss}})_Q \xrightarrow{\sim} H^\bullet(\mathcal{M}_d^{\theta-\text{ss}}) \) is an isomorphism, so that

\[ \mathcal{A}_Q^{\theta-\text{ss}} \cong \mathcal{H}_Q^{\theta-\text{ss}} \]

as graded algebras. Ultimately, this follows from the Harder–Narasimhan filtration and the fact that

\[ cl : A^\bullet(B\text{GL}_d) \xrightarrow{\sim} H^\bullet(B\text{GL}_d). \]

This begs the question: why use \( \mathcal{A}_Q^{\theta-\text{ss}} \) over \( \mathcal{H}_Q^{\theta-\text{ss}} \)?
The key tool, unavailable in cohomology, in the following.

**Theorem (Localization exact sequence)**

Let $Z \subset X$ be a closed subscheme. Then there is an exact sequence

$$A_i(Z) \to A_i(X) \to A_i(X \setminus Z) \to 0.$$ 

Apply this to the $\text{GL}_d$-schemes

$$Z = R_d \setminus R_d^{\text{st}}, \quad X = R_d, \quad X \setminus Z = R_d^{\text{st}}.$$ 

Note that

$$Z = \bigcup_{(d',d'') \in \Lambda_Q^+ \times \Lambda_Q^+} R_{d',d''} \times \text{GL}_{d',d''} \text{GL}_d$$

$$(d'+d''=d, \quad d',d'' \neq 0)$$

and

$$A_{\bullet}^{\text{GL}_d} (R_{d',d''} \times \text{GL}_{d',d''} \text{GL}_d) \simeq A_{\bullet}^{\text{GL}_{d',d''}} (R_{d',d''}) \simeq A_{\bullet}^{\text{GL}_{d''} \times \text{GL}_{d''}} (R_{d'} \times R_{d''}).$$
Theorem (Franzen–Reineke)

The kernel of the restriction map

\[ \mathcal{A}^{\theta-ss}_Q \to \bigoplus_{d \in \Lambda^+_Q} H^\bullet(\mathbf{M}^{\theta-st}_d) \]

is

\[ \sum_{(d',d'') \in \Lambda^+_Q \times \Lambda^+_Q} \mathcal{A}^{\theta-ss}_{Q,d'} \otimes \mathcal{A}^{\theta-ss}_{Q,d''} \xrightarrow{\text{mult}} \mathcal{A}^{\theta-ss}_Q. \]

In particular, there are graded vector space isomorphisms

\[ V_{Q}^{\text{prim},\theta} \otimes \mathbb{Q}[z] \simeq \mathcal{A}^{\theta-ss}_Q / (\mathcal{A}^{\theta-ss}_Q,+ \cdot \mathcal{A}^{\theta-ss}_Q,) \simeq \bigoplus_{d \in \Lambda^+_Q} H^\bullet(\mathbf{M}^{\theta-st}_d). \]

In other settings (Meinhardt–Reineke, Chen), one uses Hodge-theoretic purity arguments in place of localization sequence.
Theorem (Milnor–Moore)

Let $\mathcal{H}$ be a connected graded cocommutative Hopf algebra. Then

$$\mathcal{H} \simeq \mathcal{U}(\mathcal{V}),$$

where $\mathcal{V}$ is the Lie algebra of primitives of $\mathcal{H}$.

Here

$$\mathcal{V} = \{x \in \mathcal{H} \mid \Delta(x) = x \otimes 1 + 1 \otimes x\},$$

and the Lie bracket is

$$[x, y] = xy - yx.$$
Theorem (Davison)

The cohomological Hall algebra $\mathcal{H}_Q$ admits a coproduct (in a certain generalized sense)

$$\Delta : \mathcal{H}_Q \rightarrow \mathcal{H}_Q \otimes \mathcal{H}_Q,$$

which makes $\mathcal{H}_Q$ into a localized bialgebra.
Theorem (Davison–Meinhardt)

There is a filtration $\mathcal{P}$ on $\mathcal{H}_Q$, making $\text{gr}^\mathcal{P}(\mathcal{H}_Q)$ into a bialgebra. Moreover, there is a bialgebra isomorphism

$$\text{gr}^\mathcal{P}(\mathcal{H}_Q) \cong \mathcal{U}(V_Q^{\text{prim}} \otimes \mathbb{Q}[z]),$$

where $V_Q^{\text{prim}} \otimes \mathbb{Q}[z]$ is the Lie algebra of primitives of $\text{gr}^\mathcal{P}(\mathcal{H}_Q)$.

Presumably, this is the Harvey–Moore BPS Lie algebra. It contains only single particle BPS states, while $\mathcal{H}_Q$ is the algebra of multiparticle (BPS) states.
Surveys on DT theory