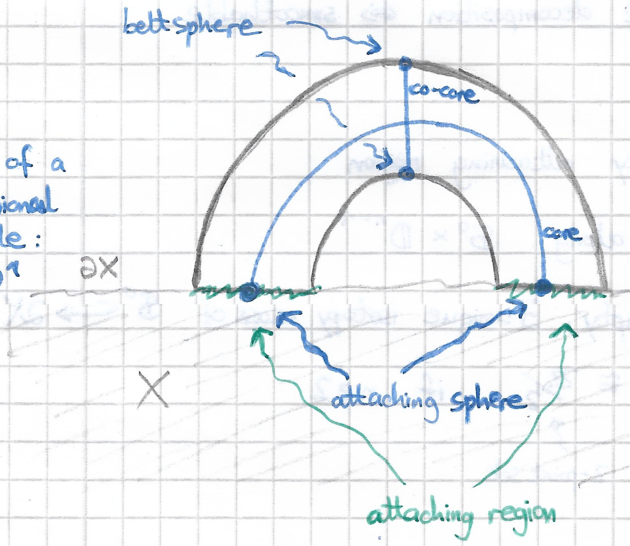
Handle decompositions

$0 \leq k \leq n$ ← n is the dim.

$\mathbb{D}^k \times \mathbb{D}^{n-k}$ — **n-dimensional k-handle** ← k is called the index of the handle ("thickened k-cell")

Anatomy of a handle:

Picture is of a 2-dimensional 1-handle: $\mathbb{D}^1 \times \mathbb{D}^1$



$\mathbb{D}^k \times \{0\}$ core

$\{0\} \times \mathbb{D}^{n-k}$ cocore

$\partial\mathbb{D}^k \times \mathbb{D}^{n-k}$ attaching region

$\partial\mathbb{D}^k \times \{0\}$ attaching sphere

$\{0\} \times \partial\mathbb{D}^{n-k}$ belt sphere

Handles attached using attaching maps

$$\varphi : \partial\mathbb{D}^k \times \mathbb{D}^{n-k} \longrightarrow \partial X \quad \text{(smooth) embedding}$$

where X is an n-manifold.

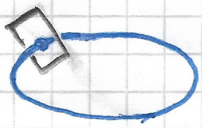
HW1: Isotopy of φ does not affect diffeomorphism type of $X \cup_\varphi h$

Then $X \cup_\varphi (\mathbb{D}^k \times \mathbb{D}^{n-k})$ is specified by

an (i) embedding $S^{k-1} \hookrightarrow \partial X$ (i.e. a knot in ∂X)

and a (ii) trivialization of its trivial normal bundle (i.e. a framing)

Ex. Suppose $k=2$



Fix some reference framing f_0

then any other framing corresponds to an element

of $\pi_{k-1}(O(n-k))$
$$\begin{array}{c} \wr\!\!\wr \text{ Gram-Schmidt} \\ GL(n-k) \end{array}$$

Always: "Smooth the corners" so that $X \cup_\varphi h$ is a smooth n-manifold

Given a compact manifold X, $\partial X = \partial_- X \amalg \partial_+ X$, a **handle decomposition** of $(X, \partial_- X)$ is an identification of X with $(\partial_- X \times I) \cup \{\text{handles}\}$

Fact: Any smooth, compact manifold pair $(X, \partial_- X)$ has a relative handle decomposition.

Aside: Any topological n-mfld. pair has a topological handle decomposition, **except** when $n=4$!

References: $n=3$ Moise, $n \geq 6$ Kirby–Siebenmann, $n=5$ Freedman–Quinn

A 4-manifold has a topological handle decomposition $\Leftrightarrow$ smoothable

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0-handles: $\mathbb{D}^0 \times \mathbb{D}^n$ has empty attaching region

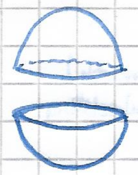
1-handles: $\mathbb{D}^1 \times \mathbb{D}^{n-1}$ attached along $\mathbb{S}^0 \times \mathbb{D}^{n-1}$

- if ∂X (compact) connected, nonempty, $\exists$ unique isotopy class of $\mathbb{S}^0 \hookrightarrow \partial X$
- framings $\underbrace{\pi_0(O(n-1)) \cong \mathbb{Z}/2}_{\text{2-point set}}$ if $n \geq 2$

framing determines whether resulting manifold orientable. i.e. if result orientable, $\exists$ unique choice of framing

Ex.: Surfaces

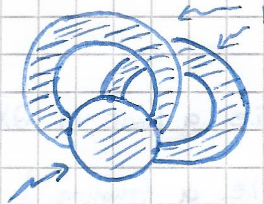
- Sphere $\mathbb{S}^2$:



$\mathbb{D}^2 \times \mathbb{D}^0$ 2-handle

$\mathbb{D}^0 \times \mathbb{D}^2$ 0-handle

- Torus:

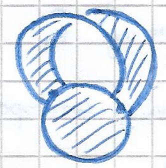


$\mathbb{D}^1 \times \mathbb{D}^1$ 1-handles

$\cup$ (2-handle) attached along S^1 boundary

0-handle $\mathbb{D}^0 \times \mathbb{D}^2$

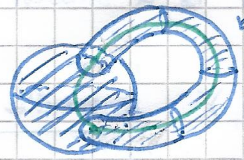
- Möbius band



different framing for the 1-handle

non-orientable!

Ex.: 3-manifolds



3-dim. 1-handle $\mathbb{D}^1 \times \mathbb{D}^2$

0-handle $\mathbb{D}^0 \times \mathbb{D}^3$

result is $\mathbb{S}^1 \times \mathbb{D}^2$

$\underbrace{(n-1)}_{\text{for } n\neq 2}$ - and n-handles have unique framings:

$k = n-1, \quad \pi_{n-2}(\overbrace{O(1)}^{=S^0}) = 0$ if $n \neq 2$

$k = n, \quad \pi_{n-1}(O(0)) = 0$

Note: n-handles attached along $\partial D^n = S^{n-1}$

For $n \leq 4$, any self-diffeo of S^{n-1} is isotopic to either identity or reflection. → Exotic spheres in higher dim.

$\Rightarrow \exists$ unique way to attach an n-handle to S^{n-1} for $n \leq 4$

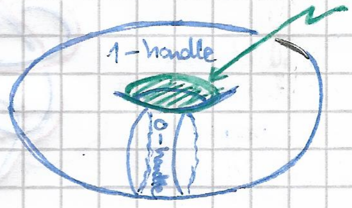
2-handles:

$$\text{framings: } \pi_1(O(n-2)) \cong \begin{cases} 0 & n \leq 3 \\ \mathbb{Z} & n = 4 \\ \mathbb{Z}/2 & n \geq 5 \end{cases}$$

In particular: D^2-bundles over S^2 correspond to integers $\mathbb{Z}$

"clutching function"
$S^1 \longrightarrow O(2)$
equator of S^2

Ex: S^3



"coin" or "plug"

$\cup$ (3-handle)

Turning handles upside down: Given a (relative) handle decomposition for $(X^n, \partial_- X)$ we can produce one for $(X, \overline{\partial_+ X})$

bar denotes opposite orientation

Every k-handle in $(X, \partial_- X)$ becomes an $(n-k)$-handle in $(X, \overline{\partial_+ X})$

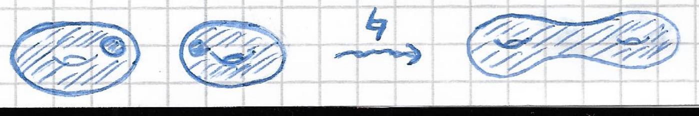
Fact (HW 1(~~c?~~ b)): If X is connected, we can assume it has a single 0-handle (if $\partial_- X = \emptyset$) or no 0-handle (if $\partial_- X \neq \emptyset$)

3-manifold topologists ~~Some people~~ call H_m a handlebody

3-manifolds (closed, orientable)

$$\underbrace{\{0\text{-handle}\} \cup \{m \text{ 1-handles}\}}_{H_m := \natural^m S^1 \times D^2} \cup \underbrace{\{k \text{ 2-handles}\} \cup \{3\text{-handle}\}}_{\natural^k S^1 \times D^2}$$

Boundary connected sum $\natural$:



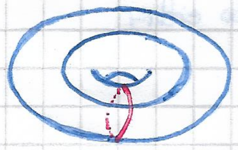
Since 3-mfld. is closed $\Rightarrow k = m$

Any 3-manifold (closed, orientable) is the union of 2 copies of H_m for some m.

This is called a <u>Heegaard decomposition.</u>

The 3-manifolds obtained from $\mathbb{S}^1 \times \mathbb{D}^2 \cup \mathbb{S}^1 \times \mathbb{D}^2$ } Genus 1 Heegaard splitting

are called the <u>Lens spaces.</u>



$\mathbb{S}^3$

$\mathbb{RP}^3$

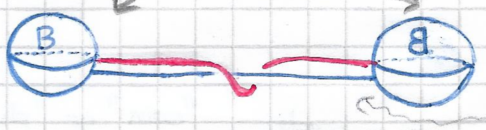
2-handles attached along these curves

In the case that the boundary is $\mathbb{S}^2$, the 3-handle attaches uniquely

<u>4-manifolds</u> (<u>Kirby diagrams</u>)

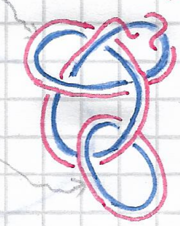
0-handle has boundary $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$

the two feet of a 1-handle



blue: attaching spheres of 2-handles

red curves specifies framing of 2-handles



⚠ Have to be careful when they are running over a 1-handle
↗ belt trick

The two balls are identified via a reflection - this is illustrated by the "B" - "B" pair of labels

If 4-mfld. is closed, oriented $\{3\text{- and } 4\text{-handles}\} \cong \natural^m \mathbb{S}^1 \times \mathbb{D}^3$

so has boundary $\#^m \mathbb{S}^1 \times \mathbb{S}^2$

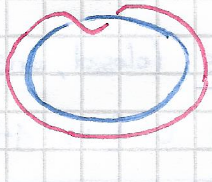
<u>Laudenbach - Poenaru:</u> Any diffeomorphism of $\#^m \mathbb{S}^1 \times \mathbb{S}^2$ extends over $\natural^m \mathbb{S}^1 \times \mathbb{D}^3$

<u>Upshot:</u> If X closed, no need to specify (draw) the 3- and 4-handles.

<u>Ex.:</u>

If we want this to represent a closed mfld., have to add $\{3\text{-handle}\} \cup \{4\text{-handle}\}$ (which we do not draw)

$\rightsquigarrow \mathbb{S}^1 \times \mathbb{S}^3$



$\rightsquigarrow \mathbb{CP}^2$ if we add a 4-handle

References for now:

[Gompf, Stipsicz: 4-manifolds and Kirby Calculus]

[Kirby: The topology of 4-manifolds]

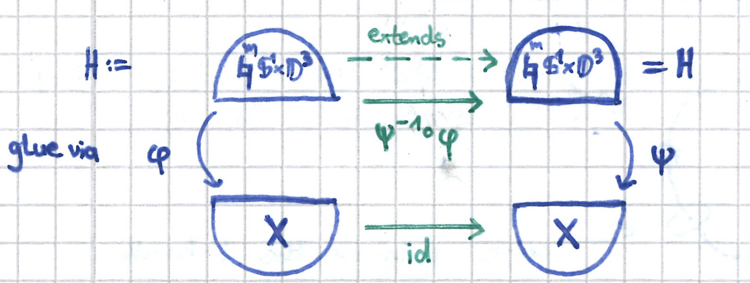
[Scorpan: The wild world of 4-manifolds] * read with caution – this book contains a lot of context, references, and intuition, but not always all the detail! Sometimes misleading

Clarification:

Laudenbach – Poenaru: Any self-diffeomorphism of $\#^m \mathbb{S}^1 \times \mathbb{S}^2$ extends to a self-diffeomorphism of $\natural^m \mathbb{S}^1 \times \mathbb{D}^3$.

Motto: "3- and 4-handles don't need to be drawn (in a diagram for a closed 4-mfld.)"

Precisely: $X \cup_\varphi H \overset{!}{\cong} X \cup_\psi H$ for all gluings φ, ψ



Need: $F: X \cup_\varphi H \longrightarrow X \cup_\psi H$

Define $F|_X := \mathrm{id}$

$F|_{\partial H} := \psi^{-1} \circ \mathrm{id} \circ \varphi : \partial H \longrightarrow \partial H$, where $\partial H = \#^m \mathbb{S}^1 \times \mathbb{S}^2$

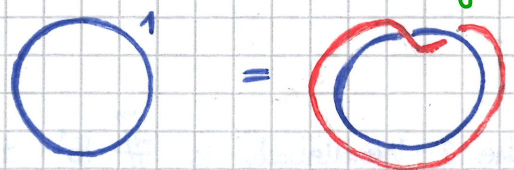
they agree on the overlap

Framings: (2-handles in a 4-manifold)

Fix a reference framing.

Framings correspond to $\pi_1(O(2)) \cong \mathbb{Z}$

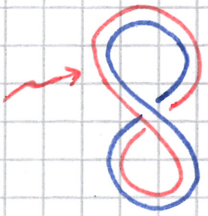
"linking number framing"



don't need to specify orientations if we require blue & red to be oriented in the same direction: flipping both orientations does not change linking numbers

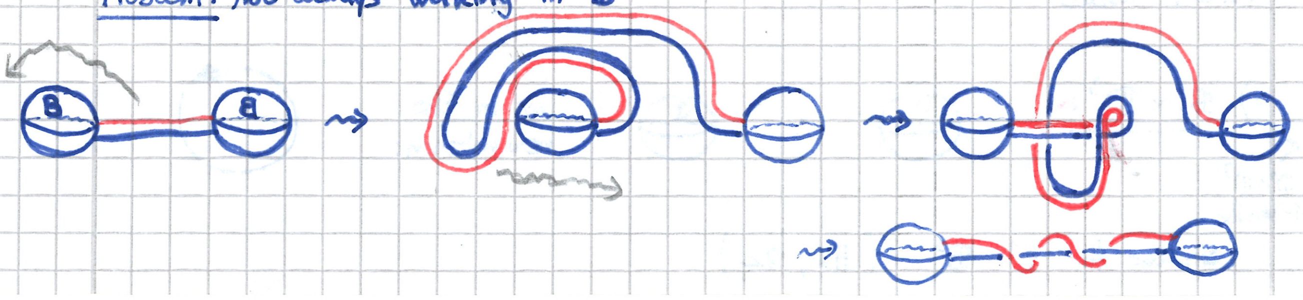
"blackboard framing":

take pushoff on the blackboard

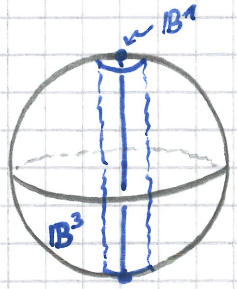


But this is not preserved under isotopies in $\mathbb{S}^3$.

Problem: Not always working in $\mathbb{S}^3$



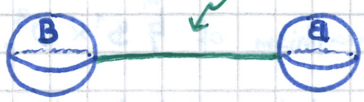
New notation for 1-handles: "A bridge is the same as an underpass"



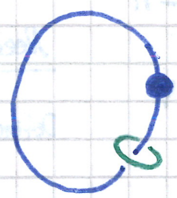
$$\mathbb{I} \times (\mathbb{B}^3 \setminus \nu \mathbb{B}^1) \cong \mathbb{I} \times (\mathbb{S}^1 \times \mathbb{D}^2)$$
$$\cong \mathbb{S}^1 \times \mathbb{D}^3$$

Old notation:

curve running over the 1-handle



New notation:



remove the "undisk" (with boundary this dotted circle) from $\mathbb{D}^4$

curve running over the 1-handle

Any (closed, smooth, oriented) 4-manifold can be represented by a link in $\mathbb{S}^3$ decorated with dots and integers.

– "dotted" components must form an unlink

> Rmk: The dot  is there so that we do not confuse 1- and 2-handles

} Is this closed?

& such that the boundary of the 2-handlebody is $\#^m \mathbb{S}^1 \times \mathbb{S}^2$ for some $m \geq 0$.

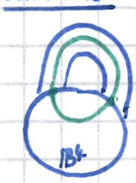
don't need this if manifold has boundary.

Examples:

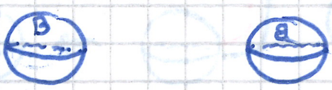
•) $\mathbb{S}^4$: $\varnothing$ ← empty diagram

has a 0-handle and a 4-handle

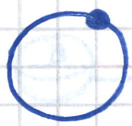
Schematic:



•) $\mathbb{S}^1 \times \mathbb{D}^3$:

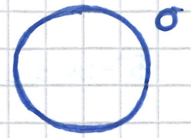


or in the new notation:

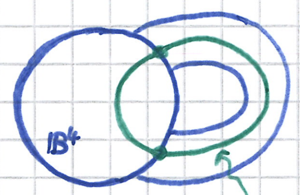


Similar for $\mathbb{S}^1 \times \mathbb{S}^3$ (HW, 2c)

·) $\mathbb{S}^2 \times \mathbb{D}^2$:



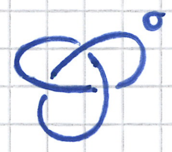
Schematic:



$\mathbb{S}^2 \times 0 \subset \mathbb{S}^2 \times \mathbb{D}^2$

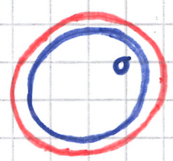
core of 2-handle
$\cup$ undisk on dotted circle

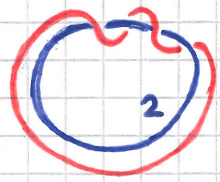
Note:

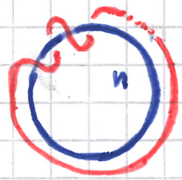


is not $\mathbb{S}^2 \times \mathbb{D}^2$

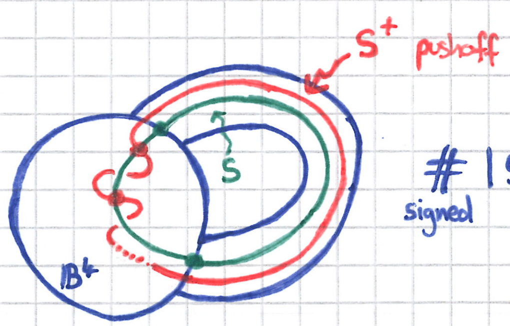
Remark:







schematic $\rightsquigarrow$



$\#$ signed $|S \cdot S^+| = n$

·) $\mathbb{S}^2 \times \mathbb{S}^2$:

$\mathbb{S}^2 = \{0\text{-handle}\} \cup \{2\text{-handle}\}$

first $\mathbb{S}^2$:	h_0	h_1
second $\mathbb{S}^2$:	j_0	j_1

$\boxed{(i\text{-handle}) \times (j\text{-handle}) = (i+j)\text{-handle}}$

$$\rightsquigarrow \mathbb{S}^2 \times \mathbb{S}^2 = \overbrace{h_0 \times j_0}^{=\mathbb{D}^2 \times \mathbb{D}^2} \cup h_0 \times j_1 \cup h_1 \times j_0 \cup h_1 \times j_1$$

$\mathbb{D}^2 \times \mathbb{S}^2$

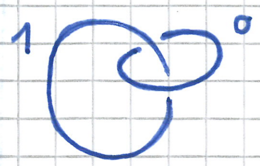
$\mathbb{S}^2 \times \mathbb{D}^2$

2-handles are attached along $\mathbb{S}^1 \times 0$ and $0 \times \mathbb{S}^1$

Namely, along the Hopf Link
(recall from previous lectures)

$= \mathbb{S}^2 \times \mathbb{S}^2$

$\cup$ (4-handle)



$= S^2 \tilde{\times} S^2$ twisted S^2-bundle over S^2

(will discuss why in future class)