Today: •) handle cancellation

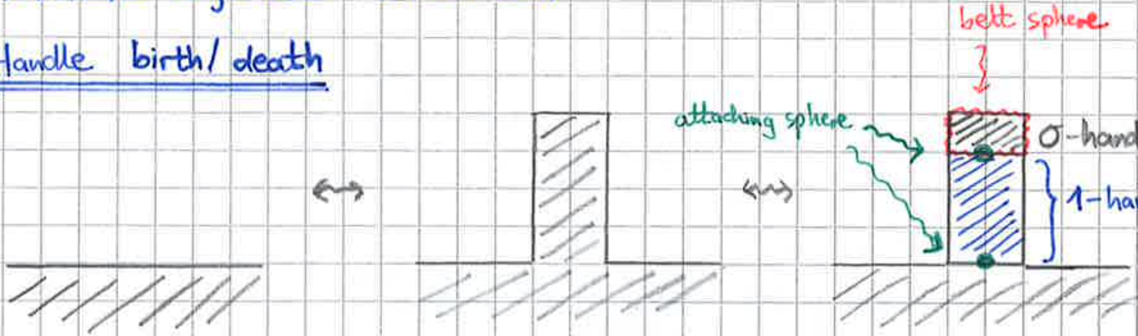
•) handle slides

•) h-cobordism theorem

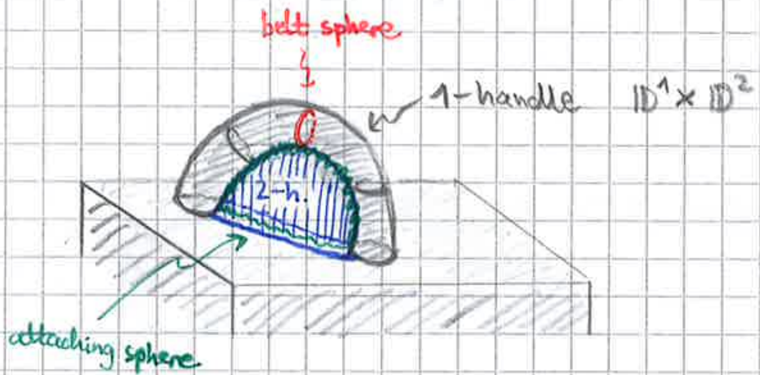
•) Poincaré conjecture (in dim ≥ 5)

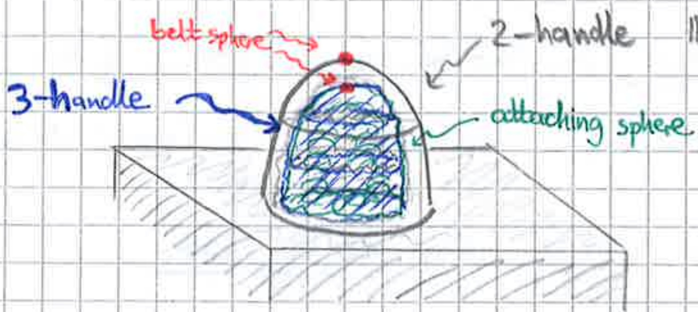
Handle birth/death

2-dim.:



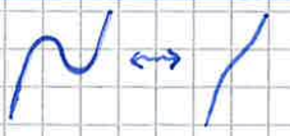
3-dim:



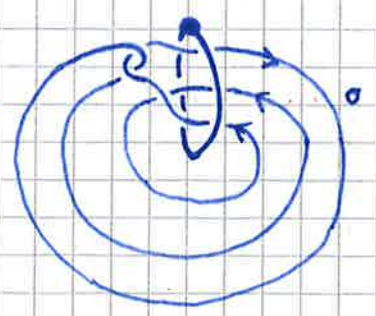


Morse cancellation Lemma: A $(k-1)$-handle $h^{(k-1)}$ can be cancelled by a k-handle $h^{(k)}$ if the attaching sphere of $h^{(k)}$ intersects the belt sphere of $h^{(k-1)}$ transversely in a single point.

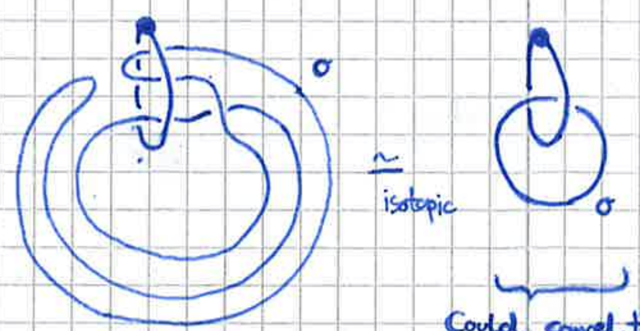
Idea:



Eg.:

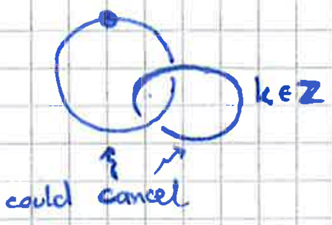


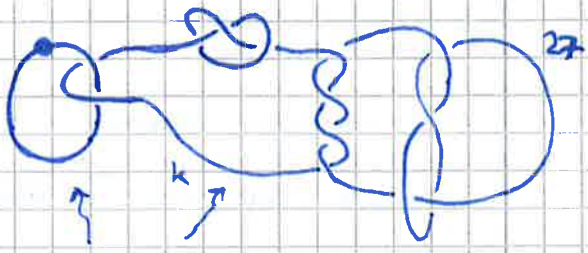
algebraic intersection number = 1
but they do not cancel!
(not obvious)



Could cancel this 2-h. with the dotted 1-h.

Framing of the 2-handles and interaction with other handles does not matter:



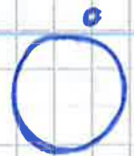


1-handle can "fall down/melt" following the 2-handle

Motto: igloo melting back into the Arctic

OR: imagine the top of a baby carriage.

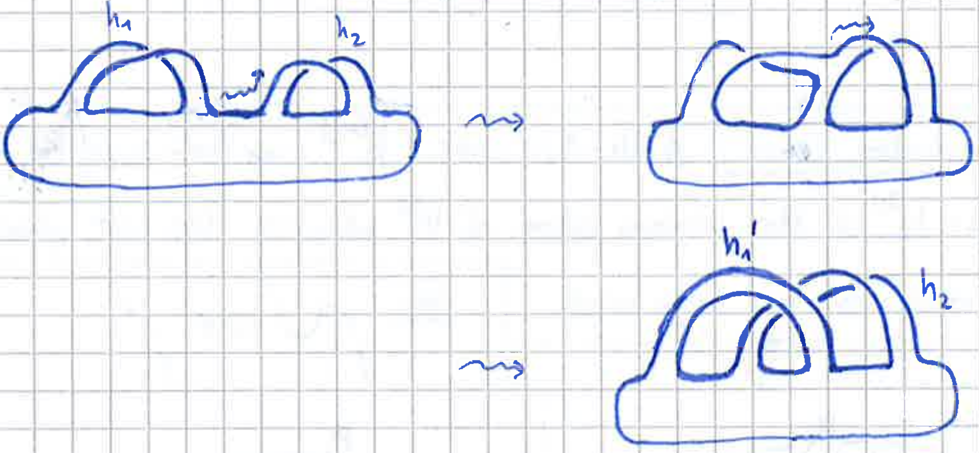
Cancelling 2-/3-handle pair:



∪ (3-handle) not drawn

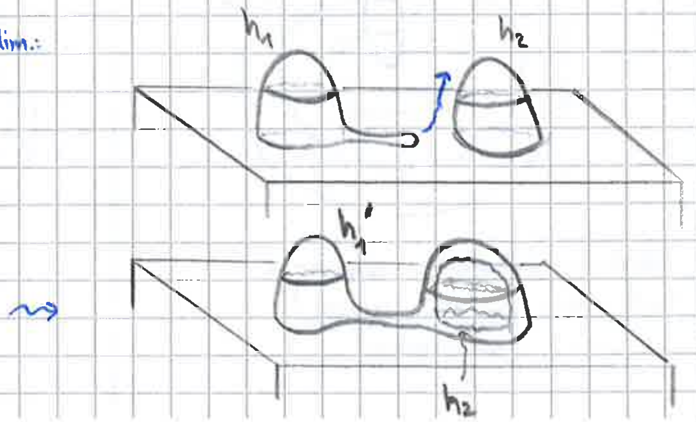
Theorem (Cerf): Any two handle decompositions of the same space are related by isotopies and handle creation/cancellation.

A particular type of isotopy: Handle slides

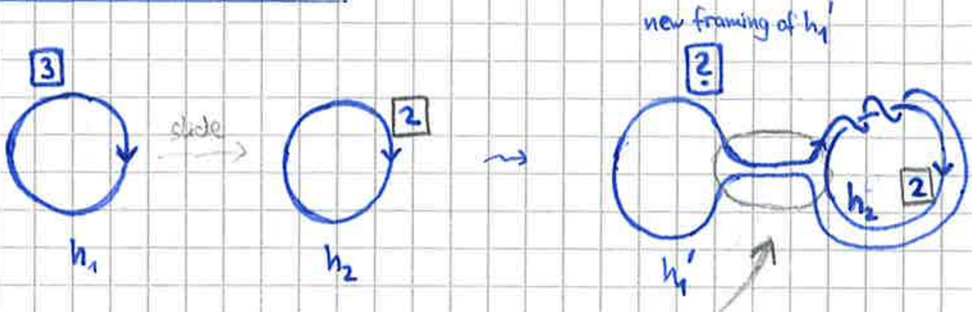


Framing and attaching sphere of h_1 changed, h_2 does not move.

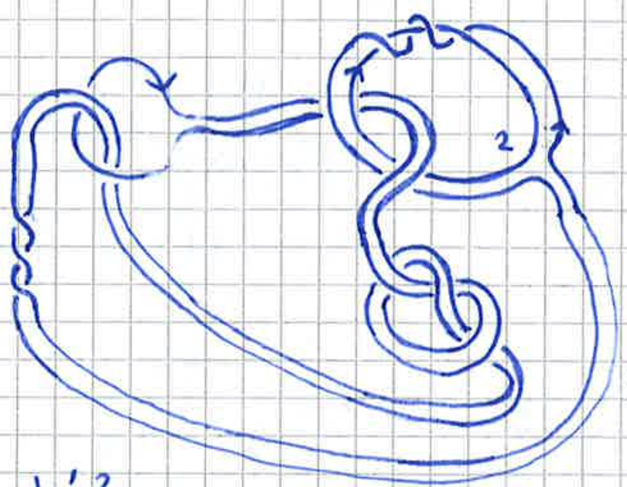
Sliding 2-handles: In 3-dim.:



Attaching circle of h_1' ?



This band can: - twist
- knot
- Link other 2-handles & 1-handles



view this as a guiding arc for the handle slide

Framing of h_1' ?

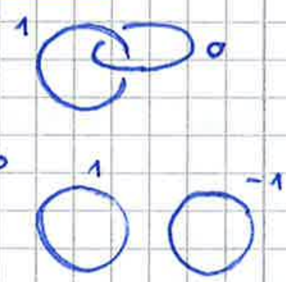
$\{\alpha_1, \dots, \alpha_m\}$ basis of $H_2(X \cup h_i)$

handle slide of h_i over h_j Leads to

$$\alpha_i' = \alpha_i \pm \alpha_j$$

$$\alpha_k' = \alpha_k \qquad \text{for } k \neq i$$

Homework: Find move from



In the case of no 1-handles:

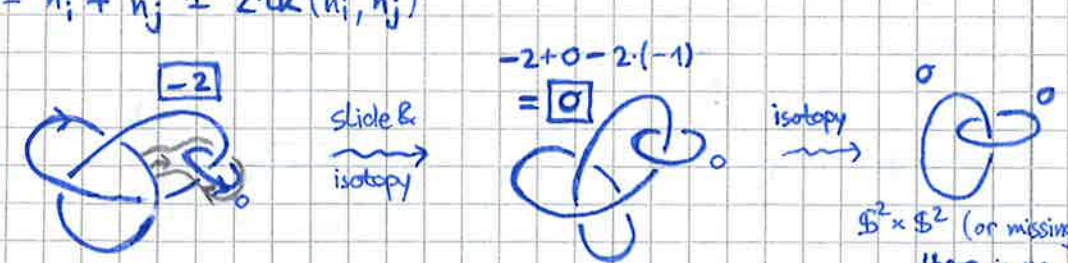
Intersection form $Q_{D^4 \cup h_i}$ is given by Linking-framing matrix

$$(\alpha_i \pm \alpha_j)^2 = \alpha_i^2 + \alpha_j^2 \pm 2 \cdot \mathrm{lk}(h_i, h_j)$$

Linking number of the attaching circles

$$n_i' = n_i + n_j \pm 2 \cdot \mathrm{lk}(h_i, h_j)$$

Ex:

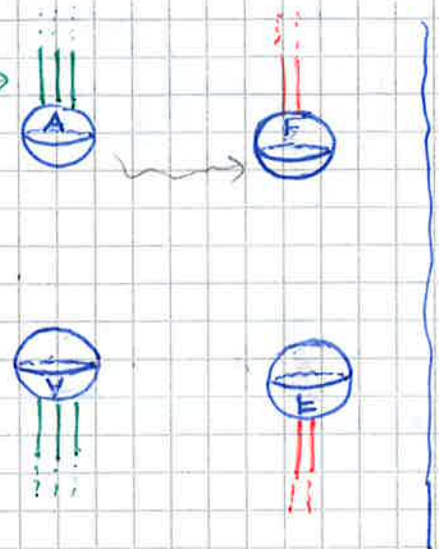


$S^2 \times S^2$ (or missing a ball if there is no 4-handle)

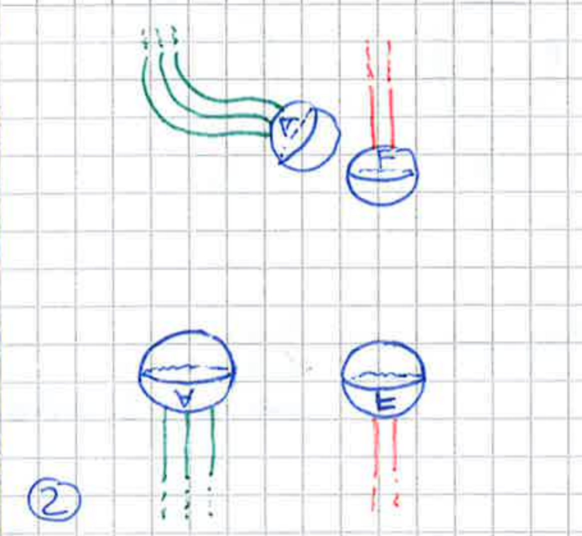
Sliding a 1-handle over another 1-handle:

In old notation:

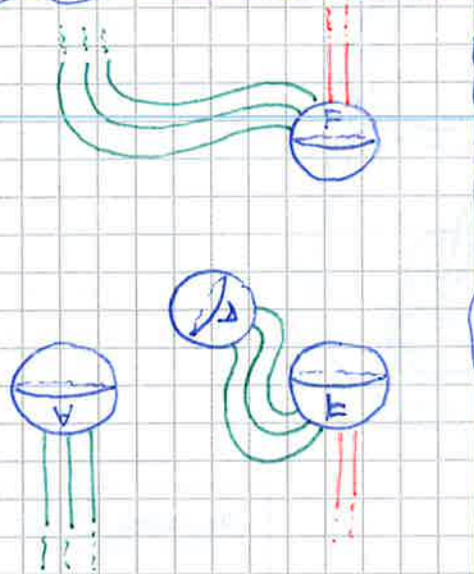
2-handles running over the 1-handle (color-coded for illustration



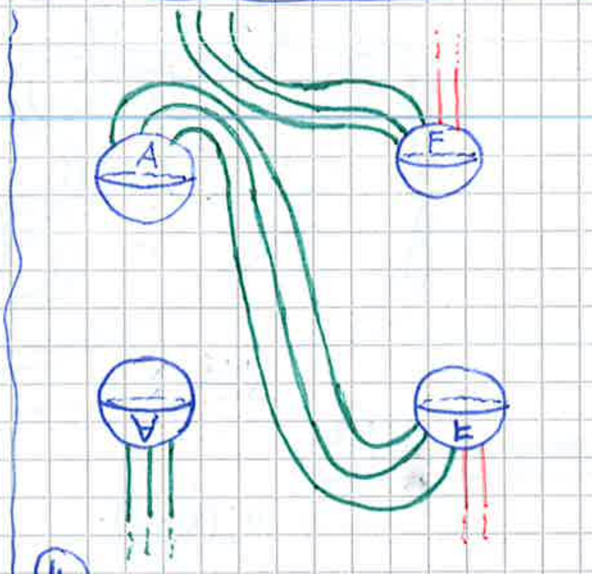
1



2

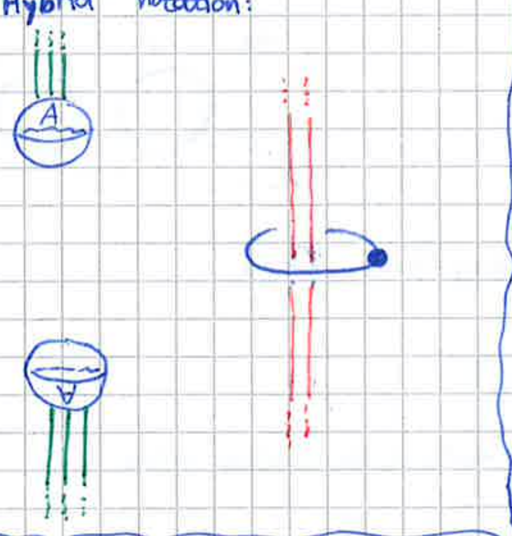


3

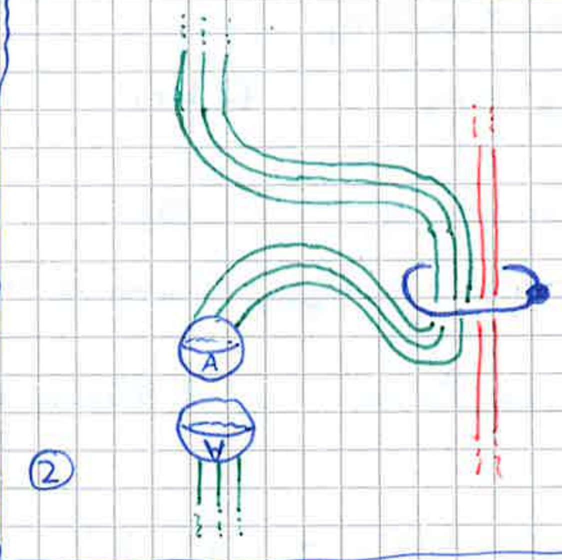


4

"Hybrid" notation:

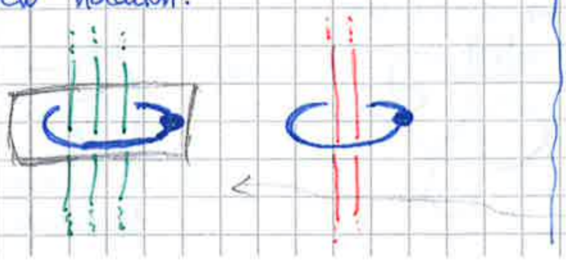


1

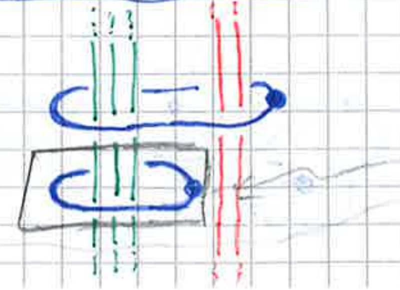


2

New notation:



1



2

⚠ This is the 1-handle doing the sliding

h - cobordisms

W^{d+1} is an **h-cobordism** if:

·) $\partial W = -M_0 \amalg M_1$

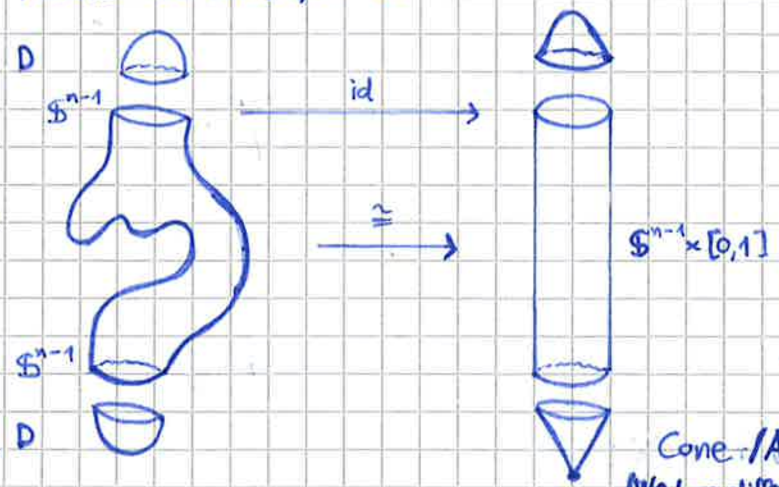
·) $M_i \hookrightarrow W$ is a homotopy equivalence for $i=0,1$

h-cobordism theorem: (Smale 1960's): Any smooth simply connected h-cobordism W^{d+1} is diffeomorphic to the product $M_0 \times [0,1]$, if $d \geq 5$.

Poincaré conjecture: Any smooth homotopy n-sphere Σ^n, $n \geq 5$, is homeomorphic to S^n.

Pf.: Remove two disks from Σ^n

Take $n \geq 6$:



Cone / Alexander trick
(We lose differentiability here)

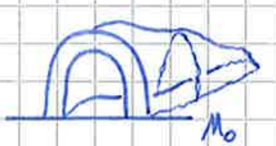
For $n=5$: $\Sigma^5 = \partial V^6$, V^6 contractible

$\Rightarrow \Sigma^5 \cong_{\text{diffeo.}} S^5$

□

Pf. of h-cobordism thm.: W is smooth $\Rightarrow \exists$ rel. handle decomp. w.r.t. M_0

- Assume no 0-handles
- "Handle trading" to remove 1-handles → replace them by 3-handles



Note: $H_*(W, M_0) = 0$

$\Rightarrow$ All handles must be cancelled algebraically

Use handle slides to do basis change ⇝ every handle either cancels or is cancelled

Suppose h^3 "cancels algebraically" h^2

$\Leftrightarrow$ attaching sphere of h^3 intersects beltsphere of h^2 algebraically once

Realize geometrically by the Whitney trick ⇝ Cobordism without handles is a cylinder! □

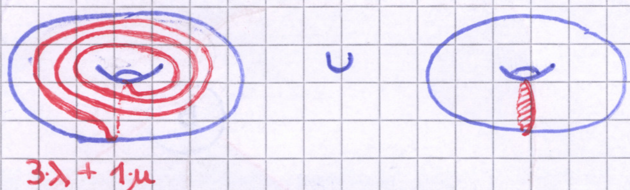
3-manifolds

- Recall we saw that any closed 3-mfld. has a Heegaard decomposition

$$M^3 = H_g \cup_{\varphi : H_g \xrightarrow{\cong} H_g} H_g$$

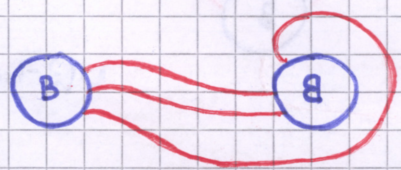
for some genus g.

Ex.: Lens space $L(3,1)$:

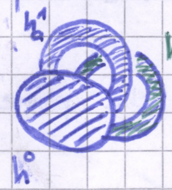


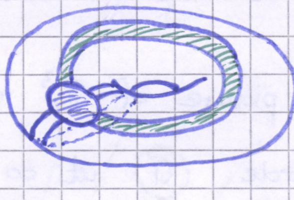
- Heegaard splittings can also be given by diagrams in the plane:

$L(3,1)$: blackboard is $\mathbb{R}^2 \cup \{\infty\} = S^2$

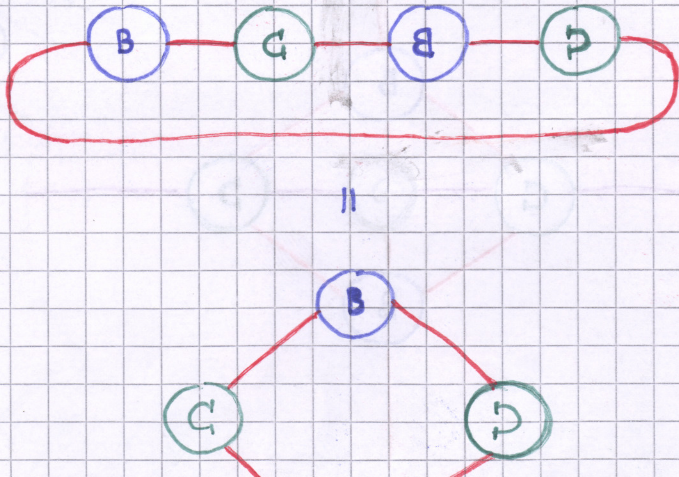


- Handle decomposition of 2-torus:

$\mathbb{T}^2 =$  $\cup$ (2-handle)

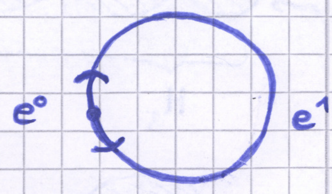


- $\mathbb{T}^2 \times \mathbb{I}$:



·) How to draw $\mathbb{T}^3$?

$$\mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{S}^1$$
$$= (h^0 \cup h^1_a \cup h^1_b \cup h^2) \times (e^0 \cup e^1)$$



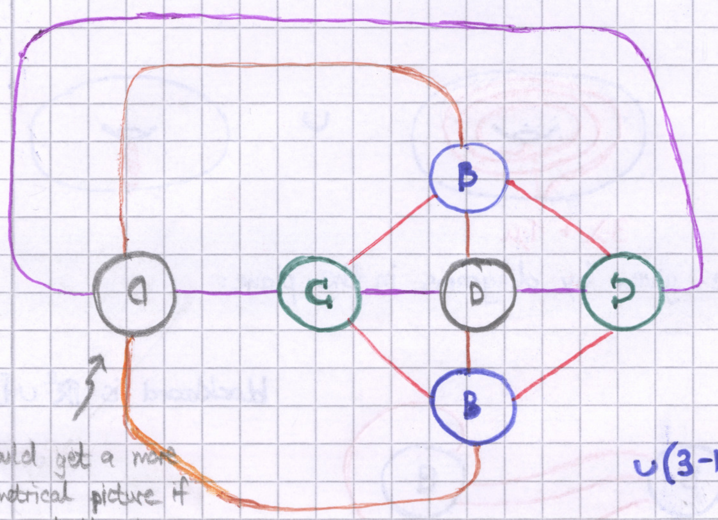
$h^0 \times e^1$ is a 1-handle

$\mathrm{core}(h^0 \times e^1) = \mathrm{core}(h^0) \times \mathrm{core}(e^1)$

$\text{attaching sphere}(h^0 \times e_1) = \partial\, \mathrm{core}(h^0 \times e^1)$

$= \mathrm{a.s.}(h^0) \times \mathrm{core}(e^1) \cup \mathrm{core}(h^0) \times \mathrm{a.s.}(e^1)$

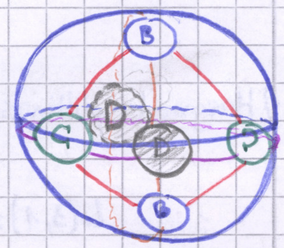
$\mathbb{T}^3 =$



would get a more symmetrical picture if we moved this to ∞

$\cup$ (3-handle)

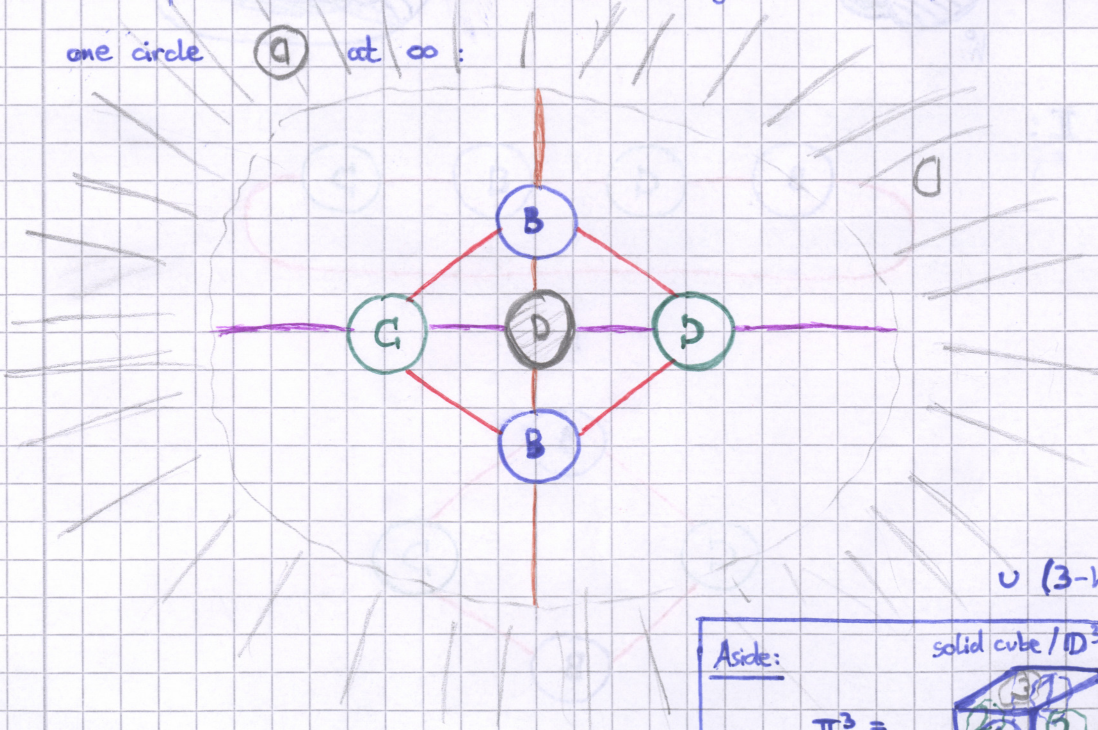
on $\mathbb{S}^2$:



$\mathrm{a.s}(h^1_a \times e^1) = \mathrm{a.s.}(h^1_a) \times \mathrm{core}(e^1) \cup \mathrm{core}(h^1_a) \times \mathrm{a.s.}(e^1)$

Better picture of $\mathbb{T}^3$ with Heegaard diagram in the plane and one circle ⓐ at ∞:

$\mathbb{T}^3 =$

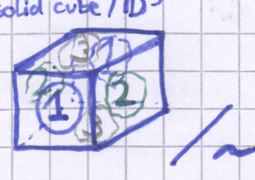


$\cup$ (3-handle)

Aside:

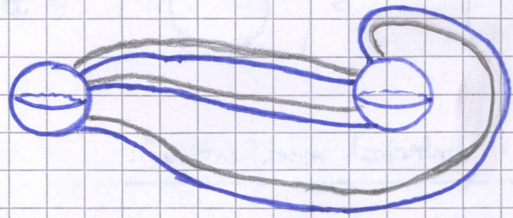
solid cube / $\mathbb{D}^3$

$\mathbb{T}^3 =$



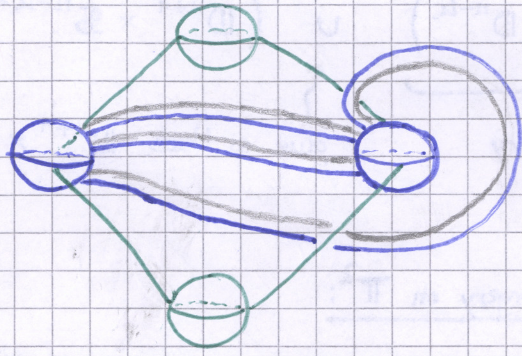
Cube with opposite faces glued

•) $L(3,1) \times \mathbb{I}$:



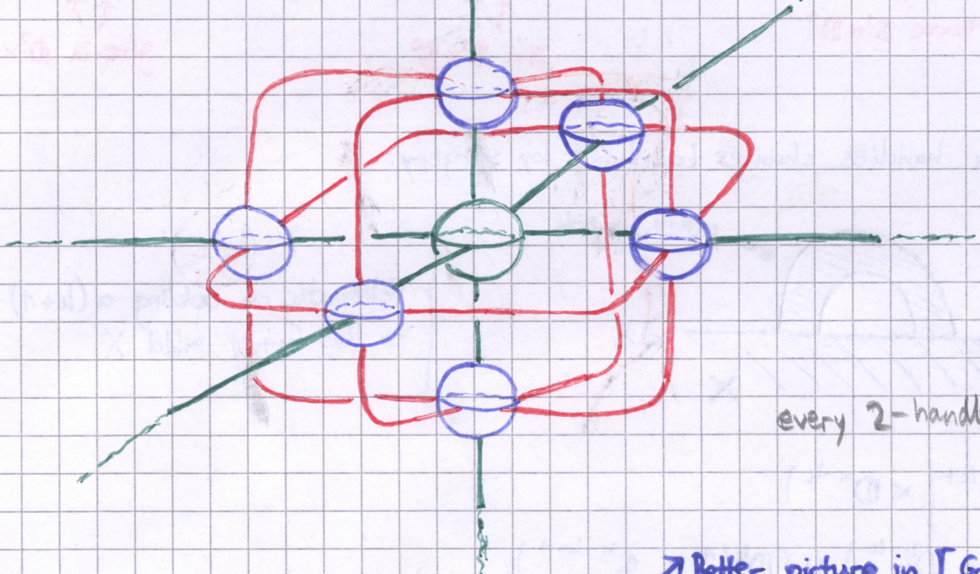
2-handle with blackboard framing

•) $L(3,1) \times \mathbb{S}^1$:



•) $\mathbb{T}^4$: one ball of the green 1-handle is at ∞:

 at ∞

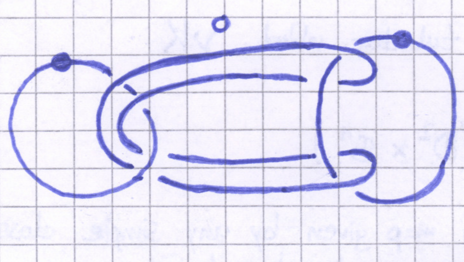


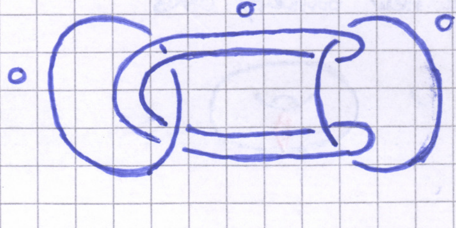
every 2-handle is 0-framed

↗ Better picture in [Gompf-Stipsicz: 4-mflds. and Kirby calculus; Fig. 4.42, p. 137]

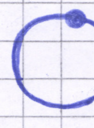
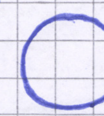
•) $\mathbb{T}^2 \times \mathbb{D}^2$ has boundary $\mathbb{T}^3$

$\mathbb{T}^2 \times \mathbb{D}^2 =$





also has boundary $\mathbb{T}^3$

$\partial\Big($  $\Big) = \partial\Big($  $\Big) = \mathbb{S}^1 \times \mathbb{D}^2$

"Surgery" or "Spherical modification":

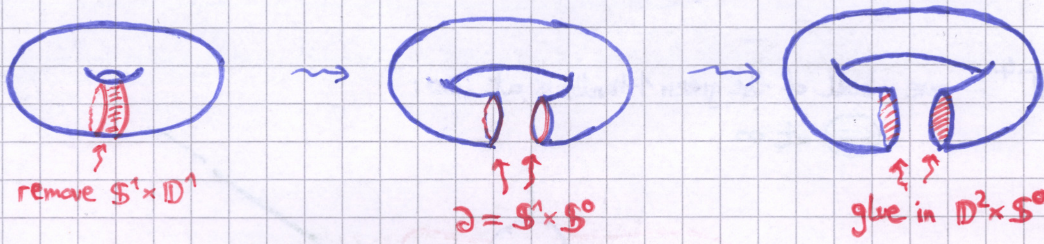
$\mathrm{Int}(M^n) \supset \mathbb{S}^k \times \mathbb{D}^{n-k}$

$$\underbrace{\left(M \setminus \mathbb{S}^k \times \mathbb{D}^{n-k} \right)}_{} \cup \left(\mathbb{D}^{k+1} \times \mathbb{S}^{n-k-1} \right)$$

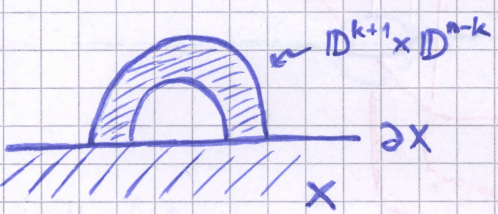
has new boundary $\mathbb{S}^k \times \mathbb{S}^{n-k-1}$

glue so that $\mathbb{D}^{k+1} \times \{*\}$ is bounded by $\mathbb{S}^k \times \{*\}$

Example of surgery on $\mathbb{T}^2$:



Adding handles changes boundary by surgery:



Schematic of adding a $(k+1)$-handle to the $(n+1)$-mfld X

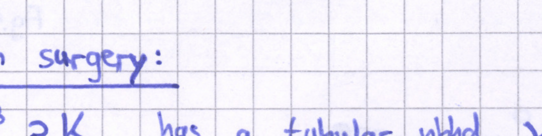
$$\partial(\mathbb{D}^{k+1} \times \mathbb{D}^{n-k})$$
$$= (\mathbb{S}^k \times \mathbb{D}^{n-k}) \cup (\mathbb{D}^{k+1} \times \mathbb{S}^{n-k-1})$$

Dehn surgery:

could use another 3-mfld as well $\rightarrow$ $\mathbb{S}^3 \supset K$ has a tubular nbhd. νK

$$(\mathbb{S}^3 \setminus \nu K) \cup (\mathbb{D}^2 \times \mathbb{S}^1)$$

gluing map given by any simple closed curve on the new boundary torus



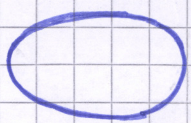
Eg.:

$\frac{1}{2}$ ← meridians
↖ (0-framed) Longitudes

is a $\mathbb{Z}HS^3$

is a "Dehn surgery diagram"

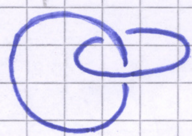
Ex.:



$\frac{p}{q}$ is a lens space $\rightsquigarrow L(p,q)$

Ex.:

$\frac{0}{1}$



$\frac{0}{1}$ $= S^3$

Fundamental theorem of 3-manifolds:

$$\frac{\left\{\begin{array}{c}\text{Framed oriented Links in } S^3\\ (\text{unknotted components, framings } \pm 1)\end{array}\right\}}{\text{isotopy} + \text{blow up/down } (+\text{ handle slides})} \xrightarrow[\text{1:1 [Kirby]}]{\text{Dehn surgery}} \frac{\left\{\begin{array}{c}(\text{connected}) \text{ closed, oriented}\\ \text{3-manifolds}\end{array}\right\}}{\text{homeo.}}$$

Surj.: [Lickorish-Wallace]

More coming up on Dehn surgery + 3 mflds + fund theorem on Thursday!

(Surgery and) Dehn surgery

Recall: Surgery is the effect on the boundary when we attach a handle

$\partial^+(X \cup_{\text{attaching sphere}} h) =$ result of doing surgery on $\partial^+ X$ along the attaching sphere

Data: $\varphi: \mathbb{S}^k \hookrightarrow M^n$ with framing f of the normal bundle

determines $\hat{\varphi}: \mathbb{S}^k \times \mathbb{D}^{n-k} \hookrightarrow M^n$

$$\text{Surgery on } (\varphi, f) := \left[M \setminus \hat{\varphi}(\mathbb{S}^k \times \operatorname{Int} \mathbb{D}^{n-k})\right] \cup_{\hat{\varphi}\restriction_{\mathbb{S}^k \times \mathbb{S}^{n-k-1}}} \left[\mathbb{D}^{k+1} \times \mathbb{S}^{n-k-1}\right]$$

Isotopy class of (φ, f) determines the result. up to diffeomorphism.

Proposition: If C is a nullhomotopic circle in M^4, then the result of surgery on M along C is either $M \# (\mathbb{S}^2 \times \mathbb{S}^2)$ or $M \# (\mathbb{S}^2 \tilde{\times} \mathbb{S}^2)$

$\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ are the two $\mathbb{S}^2$-bundles over $\mathbb{S}^2$; clutching function corresponds to an element in $\pi_1 SO(3) \cong \mathbb{Z}/2$

these two might not be distinct (depending on M)

Pf: Write $M = M \# \mathbb{S}^4$

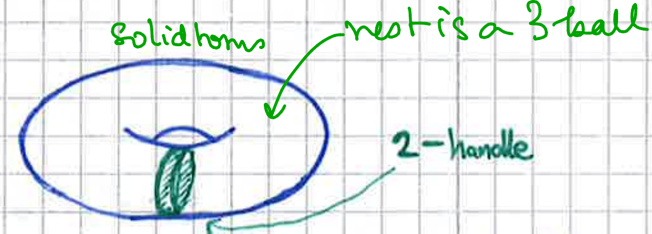
Consider $C_0 \subset \mathbb{S}^3 \subset \mathbb{S}^4$ with C_0 nullhomotopic, in particular C homotopic to C_0.

Homotopy implies isotopy for loops in a 4-manifold

$\Rightarrow C$ and C_0 are isotopic

By construction, the two different framings on C_0 transform $\mathbb{S}^4$ to $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ $\square$

intersection form: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$



Dehn surgery

$K \subset M^3$ oriented 3-manifold

Dehn surgery on M along K, according to framing φ where $\varphi: \mathbb{T}^2 \to \mathbb{T}^2$ is a diffeo.

$$M(K, \varphi) := (M \setminus \nu K) \cup_{\substack{\varphi: \mathbb{T}^2 \to \mathbb{T}^2 \\ \text{diffeo}}} \mathbb{S}^1 \times \mathbb{D}^2$$

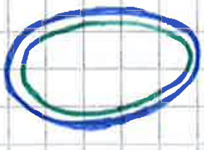
In $\mathbb{S}^3$, φ is given precisely by any pair of rel. prime integers

If K is oriented, define μ to be positive meridian
λ to be 0-framed longitude

note changing the orientat^n of K will change the orient^n of both μ & λ. So, the orientation of K is irrelevant.

Then $(p,q) = 1$, $p \cdot \mu + q \cdot \lambda$ is a unique simple closed curve in $\partial(\mathbb{S}^3 \setminus \nu K) \cong T^2$

Examples: **(a)**

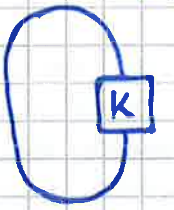
$L(0,1) =$  $\sigma = \frac{0}{1} \rightsquigarrow 0\cdot\mu + 1\cdot\lambda$

$\mathbb{S}^3 \setminus \nu(\text{unknot}) = \mathbb{S}^1 \times \mathbb{D}^2 \leftarrow$ one of the solid tori in a genus 1 Heegaard decomp. of $\mathbb{S}^3$

$= \mathbb{S}^1 \times \mathbb{S}^2$

$\mathbb{S}^1 \times \mathbb{D}^2 \qquad \cup \qquad \mathbb{S}^1 \times \mathbb{D}^2$

(b)

$L(1,0) =$  $\infty = \frac{1}{0}$

K any knot, for example  $\infty = \frac{1}{0}$

gives $\mathbb{S}^3$ (remove solid torus, and glue it back in the same way)

(c)

$L(p,q) :=$  $\frac{p}{q}$ $\qquad \mathbb{S}^3 \setminus \nu u =$ 

is a lens space with $\qquad q\cdot\mu' + p\cdot\lambda'$

$\pi_1 \cong \mathbb{Z}/p$

(d)

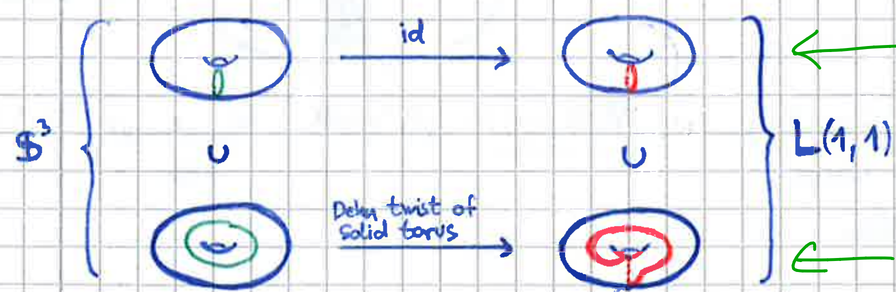
 ± 1 $= L(\pm 1, 1) = \mathbb{S}^3$

The 4-mfld. is $\mathbb{CP}^2 \setminus (4\text{-ball})$

In general:

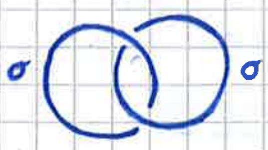
$L(p,q) \cong L(p, q + n\cdot p)$ for any n

try to prove like this



the main point is that the map on the boundary extends over the solid torus.

(e)

 $= \mathbb{S}^3$

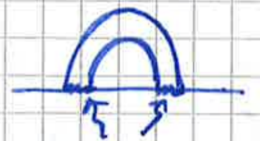
try to prove this without thinking of 4-mflds.

The 4-mfld. is $(\mathbb{S}^2 \times \mathbb{S}^2) \setminus (4\text{-ball})$

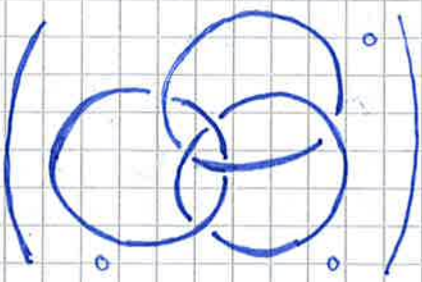
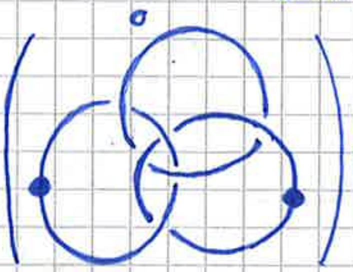
10.1.19
4-mflds.
Lecture
Aru

Big Insight: Integer-framed Dehn surgery is naturally the boundary of the 4-manifold with $\mathbb{B}^4 \cup \{2\text{-handles}\}$

(in that case, Dehn surgery = surgery)



attaching region
$\mathbb{S}^1 \times \mathbb{D}^2$

(f) ∂ () $= \partial$ () $= \mathbb{T}^3$ 3-torus

Lickorish-Wallace 1960's: Every closed, oriented 3-mfld. is the result of Dehn surgery along some link in $\mathbb{S}^3$.

The link can be chosen to have unknotted components and all the framings are ± 1.

Corollary: Any closed, oriented 3-manifold is the boundary of a simply-connected, oriented 4-mfld.

Def.:

γ

$\gamma \subset \mathbb{T}^2$ twist curve

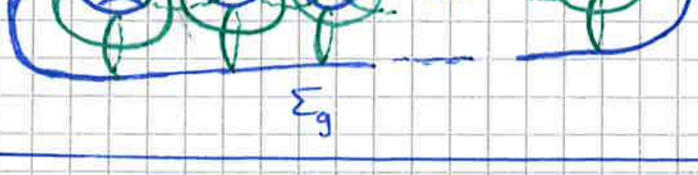
$\tau_\gamma : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$

is the Dehn twist on $\mathbb{T}^2$ along γ

Similarly on any Σ_g.

Lickorish twist theorem: Let Σ_g be a closed, orientable surface of genus g.

Any orientation-preserving diffeomorphism is isotopic to ~~a~~ some product of Dehn twists along the $3g-1$ curves below:



Σ_g

Lemma: Let H_g be the 3-dim. 1-handlebody of genus g. 

For $f: \partial H_g \longrightarrow \partial H_g$ orientation preserving diffeomorphism, there exists $\{V_i\}$, $\{V_i'\}$ each a disjoint collection of solid tori within $\operatorname{Int} H_g$ so that there is an extension

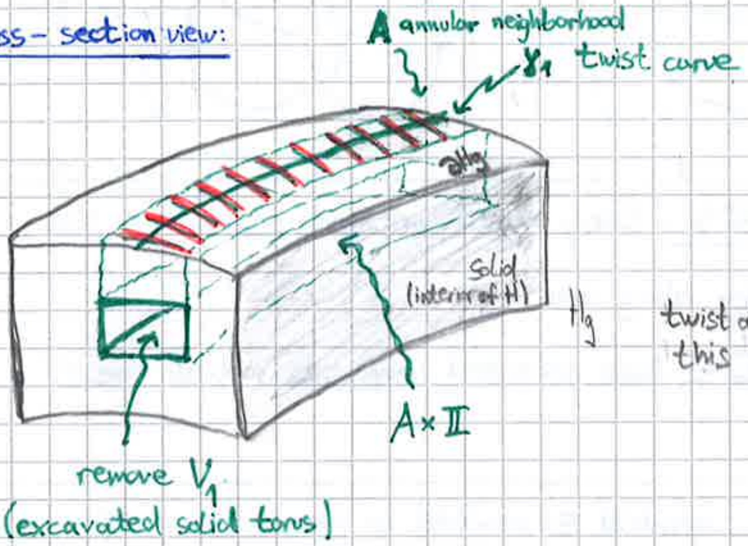
$$\bar{f}: H_g \setminus (\mathring{V}_1 \cup \mathring{V}_2 \cup \dots \cup \mathring{V}_r) \longrightarrow H_g \setminus (\mathring{V}_1' \cup \dots \cup \mathring{V}_r')$$

[Rolfsen: Knots and Links, 9.I.4, pp. 275]

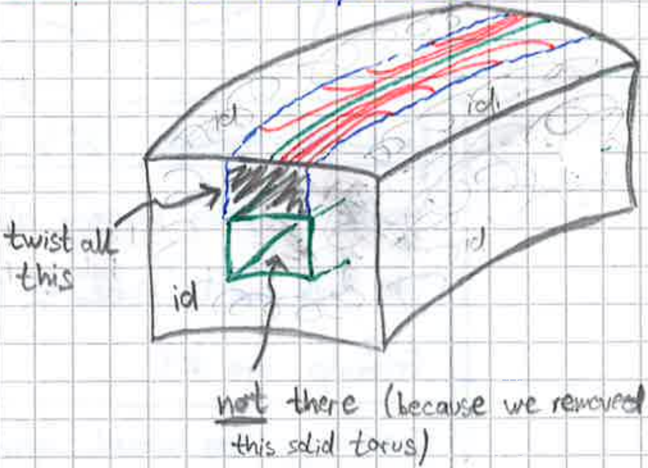
Pf.: Write $f = \tau_r \circ \dots \circ \tau_1$ as product of Dehn twists

Suppose τ_1 is a Dehn twist along γ_1, A annular neighborhood of γ_1

Cross-section view:



extend τ_1 on $A \times I$ by the product
use id everywhere else



For the other τ_i, "dig a deeper tunnel". $\square$